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Absolute continuity of some vector functions and measures	William J. Knight	737
Polynomes à valeurs entières	Paul-Jean Cahen	747
Integral means and zero distributions of Blaschke products	C. N. Linden	755
Généralisation de la formule de Riemann-Hurwitz	Ngô Van Quê	761
Taming wild simple closed curves with monotone maps	W. S. Boyd and A. H. Wright	768
Cyclic maps from suspensions to suspensions	C. S. Hoo	789
The equality $(A \cap B)^n = A^n \cap B^n$ for ideals	Robert Gilmer and Anne Grams	792
On conformally flat spaces with commuting curvature and Ricci transformations	R. L. Bishop and S. I. Goldberg	799
On Dirac's generalization of Brooks' theorem	Hudson V. Kronk and John Mitchem	805
Associated prime divisors in the sense of Krull	Richard A. Kuntz	808
On the β-construction in K-theory	C. M. Naylor	819
Certain subsets of products of metacompact spaces and subparacompact spaces are realcompact	Phillip Zenor	825
Descendingly incomplete ultrafilters and the cardinality of ultrapowers	Andrew Adler and Murray Jorgensen	830
Maximal quotient rings and S-rings	E. P. Armendariz and Gary R. McDonald	835
The number of generators of a linear p-group	I. M. Isaacs	851
Analytic Toeplitz and composition operators	James A. Deddens	859
Polynomial rings with the outer product property	A. V. Geramita	866
The strict topology in a completely regular setting: relations to topological measure theory	Steven E. Mosiman and Robert F. Wheeler	873
Right-ordered groups	A. H. Rhemtulla	891
Super-reflexive Banach spaces	Robert C. James	896
Residue free differentials and the Cartier operator for algebraic function fields of one variable	Tetsuo Kodama	905
The Hardy space H^1 on manifolds and submanifolds	Robert S. Strichartz	915
Counterexample to a conjecture on positive definite functions	James Stewart	926
Rearrangement Inequalities	Peter W. Day	930
An embedding theorem for ordered groups	Donald P. Minassian	944
Extensions of homomorphisms of parabolic subgroups	Edward N. Wilson	947
An L^p saturation theorem for splines	G. J. Butler and F. B. Richards	957
Unique factorization theorems for subalgebras of the incidence algebra	K. L. Yocom	967
On open projections of GCR algebras	Trond Digernes and Herbert Halpern	978
Right quotient rings of a right LCM domain	Raymond A. Beauregard	983
The continuity of the visibility function on a starshaped set	Gerald Beer	989

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ABSOLUTE CONTINUITY OF SOME VECTOR FUNCTIONS AND MEASURES

WILLIAM J. KNIGHT

Introduction. In the theory of vector valued functions there is a theorem which states that if a function from a compact interval I into a normed linear space X is of weak bounded variation, then it is of bounded variation. The proof uses in a straightforward way the Uniform Boundedness Principle (see [**2**, p. 60]). The present paper grew from the question of whether an analogous theorem holds for absolutely continuous functions. The answer is in the negative, and an example will be given (Theorem 7). But it will also be shown that if X is weakly sequentially complete (e.g. an L_p space, $1 \leqq p < \infty$), then a weakly absolutely continuous point function from I into X *is* absolutely continuous. The method of proof involves the construction of a countably additive set function in the standard Lebesgue-Stieltjes fashion.

The paper is divided into three parts. In Section 1 extensions of finitely additive, absolutely continuous set functions are carried out in an abstract setting. Section 2 applies this to vector valued (point) functions on the real line. This in turn is applied in Section 3 to the theory of random differential equations [**4**] to give sufficient conditions for a sample path solution to be a linear space solution as well.

The notation to be used here is standard. If A and B are sets, the symbols $A\backslash B$ denote the complement of B relative to A, even when B is not a subset of A. The symbols $A \triangle B$ denote the set $(A\backslash B) \cup (B\backslash A)$. The notation $f : A \rightarrow \mathbf{R}$ means that f is a function from A into the real numbers $\mathbf{R}$. We write "a.e." for "almost everywhere".

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1. Extension of absolutely continuous set functions.

In this section we assume throughout that $\mathscr{S}$ is a ring of sets and that λ is a countably additive, non-negative measure on $\mathscr{S}$. We assume that $\mathscr{T}$ is a subring of $\mathscr{S}$ which is dense with respect to λ; that is, the metric function $\rho(A, B) = \lambda(A \triangle B)$ makes $\mathscr{T}$ a dense subset of $\mathscr{S}$. Finally we assume that μ is a finitely additive function on $\mathscr{T}$ with values in a Banach space X.

A function ν on $\mathscr{S}$ (or $\mathscr{T}$) with values in X is called *λ-absolutely continuous* if for each $\epsilon > 0$ there is a $\delta > 0$ such that for each A in $\mathscr{S}$ (or $\mathscr{T}$), $\lambda(A) < \delta$ implies $\|\nu(A)\| < \epsilon$. The function ν is called *strongly λ-absolutely continuous*

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if for each $\epsilon > 0$ there is a $\delta > 0$ such that for any finite disjoint collection $\{A_1, \ldots, A_n\}$ of elements of $\mathscr{S}$ satisfying $\sum\lambda(A_i) < \delta$, the further inequality $\sum\|\nu(A_i)\| < \epsilon$ holds. Finally, ν is *weakly λ-absolutely continuous* if for each x^* in X^* (the topological dual of X) the composite function $x^*\nu$ is λ-absolutely continuous as a function from $\mathscr{S}$ into the scalar field of X.

THEOREM 1. *If μ is λ-absolutely continuous on $\mathscr{T}$, then μ has a countably additive extension $\bar{\mu}$ to $\mathscr{S}$ which is λ-absolutely continuous on $\mathscr{S}$. Moreover, if η is any function from $\mathscr{S}$ into X which is λ-absolutely continuous on $\mathscr{S}$ and which coincides with μ on $\mathscr{T}$, then $\eta = \bar{\mu}$.*

Proof. Since μ is finitely additive and λ-absolutely continuous, μ is uniformly continuous on $\mathscr{T}$ relative to the metric function $\rho(A, B) = \lambda(A \triangle B)$. This follows from the fact that for each $\epsilon > 0$ there is a $\delta > 0$ such that for all A and B in $\mathscr{T}$, if $\rho(A, B) < \delta$ then $\|\mu(A) - \mu(B)\| = \|\mu(A \triangle B)\| < \epsilon$. Thus by the well-known extension theorem for uniformly continuous functions into complete spaces (see [**1**, p. 23]), there exists a unique continuous extension $\bar{\mu}$ of μ to the ring $\mathscr{S}$. Moreover, $\bar{\mu}$ is uniformly continuous on $\mathscr{S}$ relative to ρ, which is to say λ-absolutely continuous. And if η is a λ-absolutely continuous extension of μ to $\mathscr{S}$, then $\eta = \bar{\mu}$.

To see that $\bar{\mu}$ is countably additive on S, let $\{A_n\}$ be a disjoint sequence in $\mathscr{S}$ whose union is also in $\mathscr{S}$. Then since λ is countably additive, $\lambda(\bigcup_{n=m+1}^{\infty} A_n) \to 0$ as $m \to \infty$, whence

$$\left\|\bar{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) - \sum_{n=1}^{m} \bar{\mu}(A_n)\right\| = \left\|\bar{\mu}\left(\bigcup_{n=m+1}^{\infty} A_n\right)\right\| \to 0.$$

It is worth noting that considerably weaker requirements may be imposed on λ, since in the proof above we used only the fact that $\lambda(A \triangle B)$ defines a metric on $\mathscr{S}$ and that λ is continuous at $\emptyset$ (i.e. if $\{B_n\}$ is a nested sequence in $\mathscr{S}$ with empty intersection, then $\lambda(B_n) \to 0$). Thus it suffices to require that λ be a non-negative function on $\mathscr{S}$ which is continuous at $\emptyset$ and which satisfies $\lambda(A) \leqq \lambda(B) + \lambda(A \triangle B)$ for all A and B in $\mathscr{S}$. (The triangle inequality follows from the last inequality.)

THEOREM 2. *If μ is strongly λ-absolutely continuous on $\mathscr{T}$, then the extension $\bar{\mu}$ of μ to $\mathscr{S}$ (Theorem 1) is strongly λ-absolutely continuous on $\mathscr{S}$.*

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that for every finite disjoint collection $\{J_1, \ldots, J_m\}$ in $\mathscr{T}$, if $\sum\lambda(J_i) < \delta$ then $\sum\|\mu(J_i)\| < \epsilon$. We shall now show that if $\{A_1, \ldots, A_n\}$ is a finite disjoint collection in $\mathscr{S}$, and if $\sum\lambda(A_i) < \delta$, then $\sum\|\bar{\mu}(A_i)\| \leqq \epsilon$. Fix $A_1, \ldots, A_n$. Take any $\alpha > 0$. We show there exists a disjoint collection $\{J_1, \ldots, J_n\}$ in $\mathscr{T}$ such that

$$\sum\lambda(J_i) < \delta \quad \text{and} \quad \sum\|\bar{\mu}(A_i) - \mu(J_i)\| < 2\alpha,$$

whence $\sum\|\bar{\mu}(A_i)\| < 2\alpha + \epsilon$, and since α is arbitrary, $\sum\|\bar{\mu}(A_i)\| \leqq \epsilon$. To construct the sets J_i we find $\beta > 0$ such that when $J \in \mathscr{T}$ and $\lambda(J) < \beta$, then $||\mu(J)|| < \alpha/n$. The proof of Theorem 1 shows that we may choose sets $B_1, \ldots, B_n$ from $\mathscr{T}$ such that $\lambda(B_i \triangle A_i) < \beta/n$ and $||\mu(B_i) - \bar{\mu}(A_i)|| < \alpha/n$. Let $B = \bigcup_{i \neq j}(B_i \cap B_j)$. Since the A_i's are mutually disjoint, $B \subset \bigcup(A_i \triangle B_i)$ and thus $\lambda(B) < \beta$. Now let $J_i = B_i \backslash B$. Note that $J_i \in \mathscr{T}$ and that the J_i's are disjoint. Moreover, $\lambda(J_i \triangle B_i) = \lambda(B_i \cap B) < \beta$, so that $||\mu(J_i) - \mu(B_i)|| < \alpha/n$, whence $\sum||\mu(J_i) - \bar{\mu}(A_i)|| < 2\alpha$.

As in the proof of Theorem 1, the foregoing proof requires less than countable additivity of λ. It suffices to assume, in addition to the continuity of λ at $\emptyset$, that λ is monotone and finitely subadditive (which implies that $\lambda(A) \leqq \lambda(B) + \lambda(A \triangle B)$ for all A and B in $\mathscr{S}$).

THEOREM 3. *If X is weakly sequentially complete and if μ is weakly λ-absolutely continuous on $\mathscr{T}$, then μ can be extended to a countably additive, λ-absolutely continuous function $\bar{\mu}$ on $\mathscr{S}$.*

Proof. Let $\mathscr{U}$ be the σ-ring generated by $\mathscr{S}$. Then λ can be extended to a countably additive $\bar{\lambda}$ on $\mathscr{U}$, with $\mathscr{T}$ dense in $\mathscr{U}$, by the usual outer measure procedure. Let $\bar{\rho}(A, B) = \bar{\lambda}(A \triangle B)$ be the corresponding metric on $\mathscr{U}$. For each $A \in \mathscr{U}$, let $\{J_n\}$ be a sequence in $\mathscr{T}$ converging under $\bar{\rho}$ to A. Then $\{J_n\}$ is a Cauchy sequence in $\mathscr{T}$. For each $x^* \in X^*$ the composite function $x^*\mu$ is $\bar{\lambda}$-absolutely continuous and thus $\bar{\rho}$-uniformly continuous, which implies that $x^*\mu(J_n)$ is a Cauchy scalar sequence. Thus $\mu(J_n)$ is weakly Cauchy. Since X is weakly sequentially complete, the proof technique for the extension of uniformly continuous functions (see [**1**, p. 23]) can be used to obtain an extension of μ to $\bar{\mu}$ on $\mathscr{U}$. Since $x^*\bar{\mu}$ obviously coincides with the extension of $x^*\mu$ to $\mathscr{U}$ obtained by applying Theorem 1 to $x^*\mu$, we see that $x^*\bar{\mu}$ is countably additive and $\bar{\lambda}$-absolutely continuous on the σ-ring $\mathscr{U}$. By the well-known theorem of Pettis (see [**1**, p. 318]), $\bar{\mu}$ is countably additive on $\mathscr{U}$. To see that $\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous on $\mathscr{U}$ we note that if $A \in \mathscr{U}$ and $\bar{\lambda}(A) = 0$, then for each $x^* \in X^*$, $x^*\bar{\mu}(A) = 0$ because $x^*\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous. Thus $\bar{\mu}(A) = 0$, which implies (see [**1**, p. 318]) that $\bar{\mu}$ is $\bar{\lambda}$-absolutely continuous on $\mathscr{U}$. Now $\bar{\mu}$ can be restricted to $\mathscr{S}$ to obtain the conclusion of the theorem.

The author is unable to see any way of weakening the requirement of countable additivity on λ without losing the λ-absolute continuity and (strong) countable additivity of $\bar{\mu}$. It is important to observe that the hypothesis of weak sequential completeness for X cannot be omitted entirely. This will be seen as a consequence of Theorem 7 in the next section.

2. Absolutely continuous functions on an interval. We now apply the preceding theorems to point functions on the real line. Let I be a compact interval in the real numbers **R**. A (point) function ξ on I with values in a Banach space X is called *absolutely continuous* on I if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all finite disjoint collections $\{(a_i, b_i)\}_{i=1}^n$ of subintervals of I, $\sum(b_i - a_i) < \delta$ implies

$$||\sum[\xi(b_i) - \xi(a_i)]|| < \epsilon.$$

The function ξ is called *strongly absolutely continuous* on I if the latter inequality in the implication can be replaced by $\sum\|\xi(b_i) - \xi(a_i)\| < \epsilon$. Finally, ξ is called *weakly absolutely continuous* on I if for each $x^* \in X^*$, the scalar valued composite function $x^*\xi$ is absolutely continuous on I. When X is finite dimensional all three forms of absolute continuity are equivalent. But when infinite dimensional spaces are admitted it is easy to find examples of functions ξ which are absolutely continuous but not strongly so (see [**2**, p. 60]). As Corollary 6 will show, it is more difficult to construct a weakly absolutely continuous function which is not absolutely continuous.

THEOREM 4. *If a point function $\xi : I \to X$ is absolutely continuous on I, then ξ generates a unique, absolutely continuous, countably additive set function ν on the Lebesgue measurable subsets of I with the property*

$$\nu((s, t]) = \xi(t) - \xi(s),$$

for $s < t$ in I. If ξ is strongly absolutely continuous, then ν is strongly absolutely continuous.

Proof. (The absolute continuity of ν is, of course, with respect to Lebesgue measure.) Let $\mathscr{T}$ denote the ring of all finite disjoint unions of subintervals of the form $(s, t]$. Then $\mathscr{T}$ is dense with respect to Lebesgue measure λ in the σ-ring $\mathscr{S}$ of Lebesgue measurable sets. Define $\mu((s, t])$ to be $\xi(t) - \xi(s)$, and extend μ in the obvious finitely additive way to $\mathscr{T}$. Clearly μ is λ-absolutely continuous on $\mathscr{T}$. Use Theorem 1 to extend μ to ν. The strong absolute continuity makes use of Theorem 2.

THEOREM 5. *Let X be weakly sequentially complete. If a point function $\xi : I \to X$ is weakly absolutely continuous, then ξ generates a unique, absolutely continuous, countably additive set function ν on the Lebesgue measurable subsets of I with the property*

$$\nu((s, t]) = \xi(t) - \xi(s)$$

for $s < t$ in I.

Proof. The proof is similar to that of Theorem 4, but we use Theorem 3 in place of Theorem 1.

COROLLARY 6. *A weakly absolutely continuous function on a compact real interval having values in a weakly sequentially complete space X is absolutely continuous.*

Proof. Let ξ be the function, $[a, b]$ the interval. Use Theorem 5 to generate the absolutely continuous set function ν. Then $\xi(t) = \xi(a) + \nu((a, t])$. Since ν is absolutely continuous, clearly so is ξ.

THEOREM 7. *There exist weakly absolutely continuous point functions, defined on compact real intervals and taking values in a Banach space, which are not absolutely continuous.*

Proof. We construct an example. Let c_0 be the B-space of all scalar sequences converging to zero; the norm of such a sequence is the maximum of the absolute values of its terms. It is well-known that c_0 is not weakly sequentially complete (see [**1**, pp. 239, 339]). Let $I = [0, 1]$. Let $\xi : I \to c_0$ be defined as follows: $\xi(0)$ is the zero vector in c_0; if $0 < t \leqq 1$, then $\xi(t) = \{\xi_n(t)\}_{n=1}^{\infty}$, where

$$\xi_n(t) = \int_0^t f_n(s)ds$$

and

$$f_n(s) = n, \text{ for } 0 \leqq s \leqq 1/2n,$$
$$f_n(s) = -n, \text{ for } 1/2n < s \leqq 1/n,$$
$$f_n(s) = 0, \text{ for } 1/n < s \leqq 1.$$

It is easy to see that ξ is not continuous at zero because $\|\xi(1/2k)\| = 1/2$ for $k = 1, 2, 3, \ldots$. Hence ξ is certainly not absolutely continuous on I.

To show that ξ is weakly absolutely continuous on I we suppose $x^* \in c_0^*$. It is well-known [**1**, p. 74] that there corresponds to x^* an absolutely convergent series $\sum_{n=1}^{\infty}\lambda_n$ such that for all $z = \{\zeta_n\}_{n=1}^{\infty}$ belonging to c_0, $x^*z = \sum_{n=1}^{\infty}\lambda_n\zeta_n$. Now

$$x^*\xi(t) = \sum_{n=1}^{\infty} \int_0^t \lambda_n f_n(s)ds.$$

Since

$$\sum_{n=1}^{\infty} \int_0^t |\lambda_n f_n(s)|ds \leqq \sum_{n=1}^{\infty} |\lambda_n|,$$

by the Fubini-Tonelli theorem we have

$$x^*\xi(t) = \int_0^t \left[\sum_{n=1}^{\infty} \lambda_n f_n(s)\right] ds,$$

which implies that $x^*\xi$ is absolutely continuous.

The preceding example shows that the hypothesis of weak sequential completeness in Theorem 3 cannot be omitted entirely; for if it could be omitted, then the proof of Corollary 6 would render every weakly absolutely continuous point function absolutely continuous.

3. Absolutely continuous stochastic processes. We now apply the results of Section 2 to stochastic processes and to the theory of random ordinary differential equations as discussed in [**4**]. A *stochastic process* on an interval I is a function, call it x, from I into the set of random variables on a probability space $(\Omega, \mathscr{F}, P)$. We will write $x(t, \omega)$ for the value of x at $t \in I$ and $\omega \in \Omega$. A sample path of x is a function $x(\cdot, \omega)$ on I for some fixed $\omega \in \Omega$. If a sample path has a derivative at $t \in I$, that derivative is denoted by $x'(t, \omega)$.

A random variable z on Ω has a p-mean if

$$\int_{\Omega} |z(\omega)|^{p} P(d\omega) < \infty.$$

When z has a p-mean it will be convenient to use the notation $z\hat{}$ to denote the equivalence class of random variables which coincide with z a.e. on Ω. (Thus $z\hat{} \in L_p(\Omega)$.) If x is a process with a p-mean at each point of I, then $x\hat{}$ denotes the obvious function from I into $L_p(\Omega)$. We shall say that a process x is L_p-*absolutely continuous* on I if x has p-means everywhere on I and $x\hat{}$ is strongly absolutely continuous (with respect to Lebesgue measure). We call x W_p-*absolutely continuous* on I if x has p-means everywhere on I and $x\hat{}$ is absolutely continuous. Since the L_p spaces, $1 \leqq p < \infty$, are known to be weakly sequentially complete (see [**1**, pp. 69, 290]), it follows from Corollary 7 that if $x\hat{}$ is weakly absolutely continuous, then x is W_p-absolutely continuous.

It is of interest to find connections between various types of absolute continuity for a stochastic process. In Theorems 8 and 9 we relate W_p- and L_p-absolute continuity to the absolute continuity of the sample paths. We note that if x is a real-valued process on $I = [a, b]$, then each absolutely continuous sample path, $x(\cdot, \omega)$, has a derivative a.e. on I, and

$$x(t, \omega) = x(a, \omega) + \int_{a}^{t} x'(\tau, \omega) d\tau.$$

(The integral is Lebesgue's.)

THEOREM 8. *If x is a real stochastic process on the compact interval $I = [a, b]$, if almost every sample path of x is monotone and absolutely continuous, and if $x(a,\cdot)$ and $x(b, \cdot)$ have p-means for some finite $p \geqq 1$, then x is W_p-absolutely continuous on I.*

Proof. Assume without loss of generality that every sample path of x is absolutely continuous and monotone. (Some paths may be increasing, the rest decreasing.) The inequality $|x(t, \omega)| \leqq |x(a, \omega)| + |x(b, \omega)|$ shows that x has a p-mean at each point of I. We shall show that $x\hat{}$ is weakly absolutely continuous on I. (As remarked above, this will show x is W_p-absolutely continuous.) Let x^* be a continuous linear functional on $L_q(\Omega)$. As is well-known, there exists a function φ on Ω with $\varphi\hat{} \in L_p(\Omega)$, $(1/p) + (1/q) = 1$, such that for all functions y on Ω with $y\hat{} \in L_p(\Omega)$,

$$x^*(y\hat{}) = \int_{\Omega} \varphi(\omega) y(\omega) P(d\omega).$$

Let $E = \{\omega \in \Omega: x(a, \omega) < x(b, \omega)\}$, $F = \Omega \backslash E$, $G = \{\omega \in \Omega : \varphi(\omega) > 0\}$, and $H = \Omega \backslash G$. Next let $D_1 = E \cap G$, $D_2 = F \cap G$, $D_3 = E \cap H$, and $D_4 = F \cap H$. The absolute continuity of the sample paths of x implies that for each $t \in I$,

$$x(t, \omega) - x(a, \omega) = \int_{a}^{t} x'(\tau, \omega) d\tau.$$

Thus

$$x^*(x^\wedge(t) - x^\wedge(a)) = \int_\Omega \varphi(\omega)[x(t, \omega) - x(a, \omega)]P(d\omega)$$
$$= \sum_{i=1}^{4} \int_{D_i} \int_a^t \varphi(\omega)x'(\tau, \omega)d\tau P(d\omega).$$

Now each of the four integrals is finite, and each has an integrand of constant sign. Moreover, each integrand is product measurable because $\varphi(\omega)x'(\tau, \omega)$ is product measurable. (Clearly it suffices to show that $x'(\tau, \omega)$ is product measurable; this is not difficult since x has continuous sample paths and for each t, $x(t, \cdot)$ is measurable on Ω, whence $x(t, \omega)$ is measurable in the pair (t, ω). Now $x'(\tau, \omega)$ is the limit a.e. on $I \times \Omega$ of (product measurable) linear combinations of $x(t, \omega)$.) Applying the Fubini-Tonelli theorem to each double integral separately and summing we get

$$x^*(x^\wedge(t) - x^\wedge(a)) = \int_a^t d\tau \int_\Omega \varphi(\omega)x'(\tau, \omega)P(d\omega).$$

This shows that the function

$$\alpha(t) = \int_\Omega \varphi(\omega)x'(t, \omega)P(d\omega)$$

is integrable on I and that

$$x^*x^\wedge(t) = x^*x^\wedge(a) + \int_a^t \alpha(\tau)d\tau.$$

By classical theorems the real function on the right side of this equation is absolutely continuous on I. Hence $x^\wedge$ is weakly absolutely continuous.

The processes of Theorem 8 are actually L_1-absolutely continuous and in fact have L_1 derivatives a.e. on I, as we shall show below. An *L_p derivative* at a point $t \in I$ of a process x with p-means in a neighborhood of t is the strong derivative of $x^\wedge$, that is, an element $x^{\wedge\prime}(t)$ of $L_p(\Omega)$ for which

$$\lim_{k\to 0}\|(x^\wedge(t + k) - x^\wedge(t))/k - x^{\wedge\prime}(t)\| = 0.$$

(See [**2**, p. 59].)

Note that while $x^{\wedge\prime}(t)$ is to be distinguished from $x'(t, \cdot)^\wedge$ (either may exist without the existence of the other), when $x^{\wedge\prime}(t)$ exists and $x'(t, \omega)$ exists for almost all $\omega \in \Omega$, then $x'(t, \cdot)$ has a p-mean and $x'(t, \cdot)^\wedge = x^{\wedge\prime}(t)$. This is because limits in p-norm and pointwise limits on Ω imply existence of limits in measure (P), and such limits are unique a.e. on Ω.

THEOREM 9. *If x is a real stochastic process on the compact interval $I = [a, b]$, if almost every sample path of x is monotone and absolutely continuous, and if $x(a, \cdot)$ and $x(b, \cdot)$ have 1-means, then x is L_1-absolutely continuous on I and has L_1 derivative $x'(t, \cdot)^\wedge$ for almost every $t \in I$.*

Proof. As in Theorem 8 we assume every sample path is monotone. As before, x has a 1-mean at each point of I. The absolute continuity of the sample paths yields

$$|x(b, \omega) - x(a, \omega)| = \left| \int_a^b x'(t, \omega) dt \right|.$$

The monotonicity yields

$$\left| \int_a^b x'(t, \omega) dt \right| = \int_a^b |x'(t, \omega)| dt.$$

Thus

$$(1) \quad \| x^\wedge(b) - x^\wedge(a) \|_1 = \int_\Omega \int_a^b |x'(t, \omega)| dt P(d\omega) = \int_a^b dt \int_\Omega |x'(t, \omega)| P(d\omega).$$

This shows that the function

$$\alpha(t) = \int_\Omega |x'(t, \omega)| P(d\omega) = \| x'(t, \cdot)^\wedge \|_1$$

is integrable on I; hence (cf. proof of Theorem 8) $x^\wedge$ is strongly absolutely continuous on I. Note moreover that

$$\int_I \| x'(t, \cdot)^\wedge \|_1 dt < \infty$$

follows from (1), and this in turn shows that $x'(t, \cdot)$ has a 1-mean for almost all $t \in I$.

We now show that $x'(t, \cdot)^\wedge$ is the L_1 derivative of x. As noted in the proof of Theorem 8, $x'(t, \omega)$ is product measurable on $I \times \Omega$. Thus [**1**, p. 196, Lemma 16(b)], $x'(t, \cdot)^\wedge$ is a strongly measurable function on I with values in $L_1(\Omega)$. Since

$$\int_I \| x'(t, \cdot)^\wedge \|_1 dt < \infty,$$

$x'(t, \cdot)^\wedge$ is Bochner-integrable [**2**, pp. 79,80]. Thus [**1**, p. 196] for all $t \in I$,

$$\int_a^t x'(\tau, \cdot) d\tau = (B) \int_a^t x'(\tau, \cdot)^\wedge d\tau,$$

in the sense that the random variable on the left is a member of the equivalence class of random variables represented by the Bochner integral on the right. This means that

$$x^\wedge(t) = x^\wedge(a) + (B) \int_a^t x'(\tau, \cdot)^\wedge d\tau,$$

and thus [**2**, p. 88] $x^\wedge$ has strong (L_1) derivative $x'(t, \cdot)^\wedge$ a.e. on I.

Since the assumptions of Theorems 8 and 9 are rather strong, it is worth pointing out that Theorem 9 fails if 1 is replaced by $p > 1$. For example if $(\Omega, \mathscr{F}, P)$ is taken to be the interval $[0, 1]$ with Lebesgue measure P, and if $I = [0, 1]$, then the process x defined by $x(t, \omega) = (t - \omega)^{1/2}$ for $\omega \leqq t$ and $x(t, \omega) = 0$ for $\omega > t$ is bounded and has monotone, absolutely continuous sample paths, but is not L_2-absolutely continuous, nor even of L_2 (strong) bounded variation on I. For if $0 \leqq s < t \leqq 1$, then

$$E|x(t, \cdot) - x(s, \cdot)|^2 = (1/2)(\sqrt{t} - \sqrt{s})(t^{3/2} - s^{3/2}) \\ + ((t - s)^2/4) \log ((\sqrt{t} + \sqrt{s})/(\sqrt{t} - \sqrt{s})) > ((t - s)^2/8) \log (t/(t-s)).$$

Thus if $0 \leqq a = t_0 < \ldots < t_n = b \leqq 1$ and $t_k - t_{k-1} = (b - a)/n$, then

$$\sum^{n} ||x(t_k, \cdot) - x(t_{k-1}, \cdot)||_2 > ((b - a)/n\sqrt{8}) \sum^{n} (\log k)^{1/2}$$

and the last expression is unbounded as $n \to \infty$.

However, if the sample path derivative of the process x has p-means a.e. on I, then even though x may not have an L_p derivative, it does have what we shall call a W_p *derivative*. A W_p derivative of x is a pseudo-derivative, that is, a function y, defined a.e. on I and having values in $L_p(\Omega)$, such that for each continuous linear functional x^* on $L_p(\Omega)$, the scalar valued composite function $x^*x^\wedge$ has derivative x^*y a.e. on I (see [**5**, p. 300]).

THEOREM 10. *Let a real process x satisfy the hypotheses of Theorem* 8. *If in addition the* (*sample path*) *derivative process x' has p-means a.e. on I, then $x'^\wedge$ is a W_p derivative for x on I.*

Proof. Using the notation of the proof of Theorem 8, we recall that

$$x^*x^\wedge(t) = x^*x^\wedge(a) + \int_a^t \alpha(\tau)d\tau.$$

By the classical theorems of Lebesgue integration, the derivative of the right side of the equation exists for almost all $t \in I$ and is equal to

$$\alpha(t) = \int_\Omega \varphi(\omega)x'(\tau, \omega)P(d\omega) = x^*(x'(t, \cdot)^\wedge).$$

Theorems 9 and 10 may be applied to the theory of random ordinary differential equations to produce relationships between the various types of solutions, as discussed in [**4**]. Let I be the compact interval $[a, b]$, and for each $\omega \in \Omega$ let $S(\omega)$ be a subset of $I \times \mathbf{R}$. ($\mathbf{R}$ is the set of real numbers.) Let $f(\cdot, \cdot, \omega)$ be a function from $S(\omega)$ into $\mathbf{R}$. Let x_0 be a random variable such that for almost all $\omega \in \Omega$, $(a, x_0(\omega)) \in S(\omega)$. We consider the random differential equation

$$(2) \qquad x' = f(t, x, \omega), \quad x(a, \omega) = x_0(\omega).$$

A function $x : I \times \Omega \to \mathbf{R}$ will be called a *sample path* (*S.P.*) *solution* of (2) if x is a stochastic process such that for almost every $\omega \in \Omega$ the following are true:

(a) $x(a, \omega) = x_0(\omega)$;

(b) for almost all $t \in I$, $(t, x(t, \omega)) \in S(\omega)$;

(c) $x(\cdot, \omega)$ is absolutely continuous on I;

(d) for almost all $t \in I$, $x'(t, \omega) = f(t, x(t, \omega), \omega)$.

Other types of solutions for (2) can also be defined. In particular, an *L_p-solution* is a stochastic process x with p-means everywhere on I satisfying (a) and (b) above and also

(c′) x is L_p-absolutely continuous on I;

(d′) $x^{\wedge\prime}(t) = f(t, x(t, \cdot), \cdot)^{\wedge}$ for almost all $t \in I$, where $x^{\wedge\prime}(t)$ denotes the L_p derivative of x at t.

If we require only that x be W_p-absolutely continuous on I and if we substitute the W_p derivative for the L_p derivative in (d′), the process x is called a *W_p-solution* of (2).

COROLLARY 11. *Let x be a sample path solution of equation* (2) *on $I = [a, b]$, and suppose that almost every sample path of x is monotone. If $x(a, \cdot)$ and $x(b, \cdot)$ have* 1-*means, then x is an L_1-solution of* (2).

Proof. Since an S.P. solution has absolutely continuous sample paths, this follows immediately from Theorem 9.

COROLLARY 12. *Let x be a sample path solution of equation* (2) *on $I = [a, b]$ and suppose that almost every sample path of x is monotone. If, for some finite $p \geqq 1$, $x(a, \cdot)$ and $x(b, \cdot)$ have p-means, and if, for almost every $t \in I$, $f(t, x(t, \cdot), \cdot)$ has a p-mean, then x is a W_p-solution of* (2).

Proof. This follows from Theorems 8 and 10.

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POLYNOMES A VALEURS ENTIERES

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Introduction. Soit A un anneau intègre de corps des fractions K, on s'interesse ici aux polynômes de $K[X]$ qui prennent sur A leurs valeurs dans A. Ils forment bien évidemment un sous anneau de $K[X]$ qui contient $A[X]$, nous le notons A_S.

Cet article reprend, dans un langage moderne, les idées de deux publications assez anciennes, dues à Polya [**3**] et Ostrowsky [**4**], en 1919. Les auteurs montraient que si A est principal, A_S est un A-module libre qui admet sur A une base formée de polynômes de degré croissant; si A est plus généralement l'anneau des entiers d'un corps de nombres ils mettaient en évidence une suite d'idéaux fractionnaires de A. En exposant le cas de l'anneau A principal, pour un anneau de valisation discrète dans le premier paragraphe, puis en généralisant aux anneaux de Dedekind dans notre second paragraphe, nous pouvons énoncer de façon plus explicite, que A_S est un A-module projectif, que l'on peut décomposer canoniquement en somme d'idéaux fractionnaires de A.

1. Anneaux de valuation discrète. L'anneau de Dedekind le plus simple est un anneau de valuation discrète: On note A un tel anneau, v la valuation (de son corps des fractions K) qui lui est associée; on suppose v normalisée et on note π une uniformisante. Si $\mathfrak{m}$ désigne l'idéal maximal de A, on définit la norme de $\mathfrak{m}$, notée N, comme étant le cardinal du corps résiduel $A/\mathfrak{m}$.

Soit $(a_0, a_1, \ldots, a_{N-1})$ un système de représentants de A modulo $\mathfrak{m}$, où $a_0 = 0$. Si N est infini, la suite est infinie; sinon on la prolonge de la façon suivante (d'après Polya [**4**]): Ecrivant un entier n sous la forme

$$n = i_0 + i_1 N + \ldots + i_h N^h$$

où i_j est un entier tel que $0 \leqq i_j \leqq N - 1$ pour tout j dans $\{0, \ldots, h\}$. On pose

$$a_n = a_{i_0} + a_{i_1}\pi + \ldots + a_{i_h}\pi^h.$$

On peut alors faire deux remarques:

Remarque 1. $(a_0, a_1, \ldots, a_{N^h-1})$ est un système de représentants modulo $\mathfrak{m}^h$.

Remarque 2. $v(a_n - a_{n'})$ est la plus grande puissance de N qui divise $n - n'$. On note $v_N(x)$ la plus grande puissance de N qui divise un entier x (en prenant garde qu'il ne s'agit pas d'une valuation si N n'est pas premier). On écrit alors:

$$v(a_n - a_{n'}) = v_N(n - n').$$

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Que N soit fini ou infini, on peut alors définir une suite infinie de polynômes:

$$f_0 = 1, f_1 = X - a_0, \ldots, f_n = (X - a_0) \ldots (X - a_{n-1}), \ldots .$$

Si on désigne par $f_n(A)$ l'idéal engendré par les valeurs $f_n(\xi)$ de f_n sur les élements ξ de A, $f_n(A)$ est une puissance de l'idéal maximal puisque A est l'anneau d'une valuation discrète; notant $S(n)$ cette puissance on définit ainsi une fonction de l'entier n (à valeurs entières). Avec ces notations nous reformulons un lemme du à Polya:

LEMME (Polya). *$S(n)$ ne depend pas de l'anneau A autrement que par la norme N de $\mathfrak{m}$ et c'est une fonction croissante de n. On a en effet la formule*

$$S(n) = \sum_{h=1}^{n} v_N(h) = \sum_{\alpha=1}^{\infty} \left[\frac{n}{N^n}\right]$$

où $\left[\frac{n}{N^\alpha}\right]$ *désigne la partie entière de* $\frac{n}{N^\alpha}$. *$f_n(a_n)$ engendre $f_n(A)$ autrement dit*:

$$v[f_n(a_n)] = S(n).$$

Si N est infini on donne un sens raisonnable aux formules du lemme en posant $S(n) = 0$. Par ailleurs, il est clair que $f_n(a_n)$ est inversible; donc $v[f_n(a_n)] = 0 = S(n)$ pour tout n.

Si N est fini, il nous suffit de montrer que

$$v[f_n(\xi)] \geqq v[f_n(a_n)] = S(n).$$

Quand ξ parcourt A, et de vérifier les formules pour

$$S(n) = v[f_n(a_n)].$$

Si $f_n(\xi) = 0$, $v[f_n(\xi)]$ est infini; donc on peut supposer que $f_n(\xi)$ n'est pas nul et que $v[f_n(\xi)]$ est un entier w; comme $(a_0, a_1, \ldots a_{N^{w+1}-1})$ forme un système de représentants modulo $\mathfrak{m}^{w+1}$ on trouve a_μ tel que

$$v[\xi - a_\mu] \geqq w + 1,$$

et donc tel que

$$v[f_n(a_\mu)] = v[f_n(\xi)] = w.$$

Notons que $f_n(a_\mu)$ n'est pas nul; donc que l'on a $\mu \geqq n$. Par ailleurs,

$$\begin{aligned} w &= v[f_n(a_\mu)] \\ &= v[(a_\mu - a_0) \ldots (a_\mu - a_{n-1})] \\ &= \sum_{h=0}^{n-1} v(a_\mu - a_h) \\ &= \sum_{h=0}^{n-1} v_N(\mu - h) \end{aligned}$$

(d'après la Remarque 1). Mais alors

$$w = \sum_{h=0}^{\mu} v_N(h) - \sum_{h=0}^{\mu-n} v_N(h).$$

Nous laissons au lecteur la vérification purement arithmétique de l'égalité pour tout entier μ

$$(*) \qquad \sum_{h=0}^{\mu} v_N(h) = \sum_{\alpha=1}^{\infty} \left[\frac{\mu}{N^\alpha}\right].$$

On a donc

$$\begin{aligned} v[f_n(\xi)] &= w \\ &= \sum_{\alpha=1}^{\infty} \left[\frac{\mu}{N^\alpha}\right] - \sum_{\alpha=1}^{\infty} \left[\frac{\mu - n}{N^\alpha}\right] \\ &= \sum_{\alpha=1}^{\infty} \left(\left[\frac{\mu}{N^\alpha}\right] - \left[\frac{\mu - n}{N^\alpha}\right]\right) \\ &\geqq \sum_{\alpha=1}^{\infty} \left[\frac{n}{N^\alpha}\right]. \end{aligned}$$

Les formules pour $v[f_n(a_n)]$ se déduisent de ce calcul et de l'égalité (*) en remplaçant partout μ par n.

Comme les polynômes $f_0, f_1, \ldots, f_n, \ldots$ sont de degrés croissants ils forment une base du K-espace vectoriel $K[X]$; ainsi tout polynôme P de $K[X]$ de degré n peut s'écrire de façon unique.

$$P = \lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n \quad (\lambda_i \in K, i \in \{0, \ldots, n\}).$$

Proposition 1. *Avec les notations précédentes, un polynôme P (de degré n) est dans A_S si et seulement si*

$$v(\lambda_i) \geqq -S(i) \quad (i \in \{0, \ldots, n\}).$$

Si on note I_n l'idéal fractionnaire

$$\mathfrak{m}^{-S(n)} = [f_n(A)]^{-1}$$

on a donc décomposé A_S en somme directe des idéaux $I_0, I_1, \ldots, I_n, \ldots$ selon les polynômes $f_0, f_1, \ldots, f_n, \ldots$.

Si d'abord $v(\lambda_i) \geqq -S(i)$, alors pour tout ξ dans A, $v[\lambda_i f_i(\xi)] \geqq 0$; donc $\lambda_i f_i$ est dans A_S, de même la somme $\lambda_0 f_0 + \ldots + \lambda_n f_n$ avec ces hypothèses.

Inversement, si P est dans A_S, alors $P(a_0)$ est dans A; mais alors λ_0 est dans A (car $f_i(a_0) = 0$ si $i > 0$) et $v[\lambda_0] \geqq -S(0) = 0$. Si par récurrence, les inégalités sont vérifiées pour $\lambda_0, \lambda_1 \ldots \lambda_{j-1}$, alors

$$Q_j = P - \lambda_0 f_0 - \lambda_1 f_1 \ldots - \lambda_{j-1} f_{j-1}$$

est dans A_S (d'après la première partie de cette preuve) et ainsi

$$Q_j(a_j) = \lambda_j f_j(a_j)$$

est dans A; soit

$$v(\lambda_j) \geqq -v[f_j(a_j)] = -S(j).$$

Comme les idéaux I_n sont tous principaux, A_S est un A-module libre; les polynômes $\pi^{-S(n)} f_n$ forment une base de A_S. Si N est infini, les idéaux I_n sont tous égaux à A et A_S est égal a $A[X]$.

2. Anneaux de Dedekind. Soit A un anneau de Dedekind de corps des fractions K. On note v la valuation correspondant à un idéal maximal $\mathfrak{m}_v$ de A, A_v l'anneau de valuation discrète associé (A_v est le localisé $A_{\mathfrak{m}_v}$), N_v la norme de $\mathfrak{m}_v$, et $S_v(n)$ la fonction de n (définie au § 1) sur l'anneau A_v.

Si I est un idéal fractionnaire de A, on désigne par $v(I)$ la puissance de $\mathfrak{m}_v$ dans la décomposition de I en produits d'idéaux maximaux.

Proposition 2. *Soit P un polynôme de $K[X]$ de degré n. On note $v(P)$ la plus petite des valuations des coéfficients de P et $P(A)$ le sous-A module de K engendré par les valeurs de P sur A.*

Alors $P(A)$ est un idéal fractionnaire et on a l'inégalité, pour toute valuation v,

$$v(P) \leqq v[P(A)] \leqq S_v(n) + v(P).$$

Si d est le produit des dénominateurs des coefficients de P, dP est dans $A[X]$; donc $dP(A) \subset A$ et ainsi $P(A)$ est un idéal fractionnaire. D'ailleurs, il est clair que $v(P) \leqq v[P(A)]$. Enfin, si on choisit une uniformisante π_v de v,

$$Q = \pi_v^{-v[P(A)]}P$$

est tel que $Q(A) \subset A_{\mathfrak{m}_v}$. Dans un article précédent [**2**] on a montré qu'alors $Q(A_{\mathfrak{m}_v}) \subset A_{\mathfrak{m}_v}$; donc $Q \in (A_{\mathfrak{m}_v})_S$ et ainsi on a

$$\pi_v^{-v[P(A)]}P = \lambda_0 f_{0,v} + \lambda_1 f_{1,v} + \ldots + \lambda_n f_{n,v}$$

où les polynômes $f_{i,v}$ sont construits pour A_v comme dans le paragraphe précédent (en particulier, $f_{i,v}$ est dans $A_v[X]$), et on a aussi

$$v(\lambda_i) \geqq -S_v(i) \geqq -S_v(n).$$

En définitive,

$$P = \sum_{i=0}^{n} \pi_v^{v[P(A)]} \lambda_i f_{i,v};$$

donc

$$v(P) \geqq v[P(A)] - S_v(n).$$

En analogie avec le paragraphe précédent on cherchera une suite de polynômes $f_0, f_1, \ldots, f_n, \ldots$ de degré croissant de telle sorte que tout polynôme P de $K[X]$ peut s'écrire de façon unique

$$P = \lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n \quad (\lambda_i \in K, i \in \{0, \ldots, n\}),$$

et on posera:

Définition. A_S est décomposé en somme d'idéaux fractionnaires $I_0, I_1, \ldots, I_n, \ldots$ selon des polynômes $f_0, f_1, \ldots, f_n, \ldots$ s'il est équivalent d'avoir:

(i) $P \in A_S$; ou

(ii) $\lambda_j \in I_j$ ($j \in \{0, \ldots, n\}$).

Notons tout de suite que les polynômes $f_0, f_1, \ldots, f_n, \ldots$ avec cette definition ne sont pas nécessairement dans A_S. Montrons néanmoins d'abord qu'il éxiste une telle décomposition avec des polynômes unitaires de $A[X]$.

LEMME. *Dans un anneau de Dedekind A, pour tout entier n il n'y a qu'un nombre fini de valuations v telles que $N_v \leqq n$.*

Ou bien A est un corps fini, mais on écarte ce genre de trivialité dans notre problème, ou bien il éxiste x non nul dans A tel que $x^{n-1} - 1$ n'est pas nul, mais x ou x^{n-1} est dans tout idéal m_v tel que $N_v = n$, ces idéaux sont donc en nombre fini.

On peut donc définir une suite d'idéaux fractionnaires par la condition: $v(I_n) = -S_v(n)$ pour toute valuation v de K; on est en effet assuré que $S_v(n)$ est nul pour presque tout v.

On utilise le théorème d'approximation pour définir la suite de polynômes $f_0, f_1, \ldots, f_n, \ldots$: Si, en effet, $a_{0,v}, a_{1,v}, \ldots, a_{n,v}, \ldots$ est la suite d'éléments, que l'on construit au § 1, relative à l'anneau de valuation A_v, telle que l'on a la suite de polynômes de $(A_v)_S$:

$$f_{0,v} = 1, f_{1,v} = (X - a_{0,v}), \ldots, f_{n,v} = (X - a_{0,v})(X - a_{1,v}) \ldots (X - a_{n-1,v}).$$

On sait trouver dans A des élements $\beta_{i,n}$ tels que

$$v[\beta_{i,n} - a_{i,v}] > S_v(n) (i \in \{0, \ldots, n\})$$

pour les valuations v (en nombre fini) telles que

$$S_v(n) \neq 0.$$

Si on pose

$$f_0 = 1, \ldots, f_n = (X - \beta_{0,n}) \ldots (X - \beta_{n-1,n})$$

on a

$$v[f_{n,v} - f_n] > S_v(n)$$

et

$$v[f_n(A)] = v[f_n(\beta_{n,n})] = v[f_{n,v}(a_{n,v})] = S_v(n).$$

Pour les valuations v où $S_v(n)$ n'est pas nul, mais d'aprés la Proposition 2, si $S_v(n)$ est nul

$$0 = v[f_n] \leqq v[f_n(A)] \leqq v(f_n) + S_v(n) = v(f_n) = 0$$

(puisque f_n est unitaire dans $A[X]$). Donc encore

$$v[f_n(A)] = S_v(n).$$

En conclusion

$$f_n(A) = I_n^{-1}$$

THÉORÈME 1. *Avec les notations précédentes, A_S est décomposé en somme d'idéaux fractionnaires $I_0, I_1, \ldots, I_n, \ldots$ selon les polynômes $f_0, f_1, \ldots, f_n, \ldots$.*

On va montrer ce théorème en même temps que le suivant qui établit à quelle condition une suite de polynômes permet la décomposition de A_S en sommes d'idéaux:

THÉORÈME 2. *Soit $P_0, P_1, \ldots, P_n, \ldots$ une suite de polynômes de $K[X]$, telle que P_n est de degré n. On note x_n le coefficient directeur de P_n; alors les conditions suivantes sont équivalentes:*

(i) *Pour toute valuation v, et tout entier n, on a:*

$$v[P_n(A)] = v(x_n) + S_v(n).$$

(ii) *A_S se décompose en une somme d'idéaux $J_0, J_1, \ldots, J_n, \ldots$ sur les polynômes $P_0, P_1, \ldots, P_n, \ldots$ De plus on a alors les égalités:*

$$v(x_n) = v(P_n)$$

(en particulier si P_n est unitaire il est dans $A[X]$);

$$J_n = x_n^{-1} I_n$$

(en particulier, si P_n est unitaire $J_n = I_n$).

Il est d'abord clair que (i) entraine l'égalité $v(x_n) = v(P_n)$ puisque d'une part, on a $v(x_n) \geqq v(P_n)$ par définition de $v(P_n)$ et d'autre part

$$v[P_n(A)] = v(x_n) + S_v(n) \leqq v(P_n) + S_v(n)$$

(Proposition 2). Il est clair aussi que la suite $f_0, f_1, \ldots, f_n, \ldots$ du Théorème 1 vérifie la condition (1).

(i) $\Rightarrow$ (ii). Soit donc une suite de polynômes $P_0, P_1, \ldots, P_n, \ldots$ vérifiant la condition (i). Si *on pose* $J_n = x_n^{-1} I_n$ on a

$$v[J_n] = -v(x_n) - S_v(n);$$

donc $J_n^{-1} = P_n(A)$, pour tout n. Si λ_n est dans J_n, $\lambda_n P_n$ est donc dans A_S et de même une somme

$$\lambda_0 P_0 + \lambda_1 P_1 + \ldots + \lambda_n P_n$$

où chaque λ_i est dans J_i (i variant de 1 a n).

Si par ailleurs P est un polynôme de degré n dans A_S et $P = \lambda_0 P_0 + \ldots + \lambda_n P_n$, le coéfficient directeur de P provient uniquement de P_n et c'est $\lambda_n x_n$. On a donc

$$v[\lambda_n x_n] \geqq v(P) \geqq -S_v(n) + v[P(A)]$$

et puisque $P(A) \subset A$, alors

$$v(\lambda_n) \geqq -v(x_n) - S_v(n) = v(J_n)$$

et λ_n est dans J_n. Grâce à la premiere partie du raisonnement on conclut que $P - \lambda_n P_n$ est aussi dans A_S; donc en reprenant l'argument que λ_{n-1} est dans J_{n-1}, on poursuit sans peine. . . .

(ii) $\Rightarrow$ (i). Inversement, supposons a priori que A_S se décompose en une suite d'idéaux $J_0, J_1, \ldots, J_n, \ldots$ selon les polynômes $P_0, P_1, \ldots, P_n, \ldots$ Alors pour tout élement α de J_n, αP_n est dans A_S; donc

$$\alpha P_n = \lambda_0 f_0 + \lambda_1 f_1 + \ldots + \lambda_n f_n$$

où $\lambda_k \in I_k$ ($k \in \{0, \ldots n\}$) et $f_0, f_1, \ldots, f_n, \ldots$ est la suite de polynômes du Théoreme 1. En regardant les termes de degré n on a $\alpha x_n = \lambda_n$ d'où on tire $J_n x_n \subset I_n$.

Inversement, pour tout β de I_n, βf_n est dans A_S et

$$\beta f_n = \lambda_0 P_0 + \lambda_1 P_1 + \ldots + \lambda_n P_n$$

où $\lambda_k \in J_k$, $(k \in \{0, \ldots, n\})$ d'où on tire $\beta = \lambda_n x_n$ et $I_n \subset J_n x_n$.

En définitive

$$J_n = I_n x_n^{-1}$$

Comme, pour tout α de f_n, αP_n est dans A_S, $\alpha P_n(A)$ est inclus dans A; donc $P_n(A)$ est inclus dans $J_n^{-1} = x_n I_n^{-1}$:

$$v[P_n(A)] \geqq v(x_n) + S_v(n).$$

Inversement,

$$v[P_n(A)] \leqq v(P_n) + S_v(n) \leqq v(x_n) + S_v(n)$$

(Proposition 2).

On voit que, si l'on impose aux polynômes P_n d'être unitaires, la suite des idéaux $I_0, I_1, \ldots, I_n, \ldots$ est canonique. Donnons en deux définitions intrinsèques: I_n est l'idéal fractionnaire des coéfficients directeurs des polynômes de A_S de degré n. Pour la seconde définition, faisons d'abord une remarque.

Remarque. Si on note $A_S^{(n)}$ le sous A-module de A_S formé des polynômes de A_S de degré inférieur ou égal à n, $A_S^{(n)}$ est projectif de rang $(n + 1)$ puisqu'on peut l'identifier à la somme directe:

$$A_S^{(n)} \cong I_0 \oplus \ldots \oplus I_n$$

grâce à la décomposition selon $f_0, f_1, \ldots, f_n$.

Comme la suite des idéaux $I_0, I_1, \ldots, I_n, \ldots$ est croissante, il est facile de vérifier que I_n est l'inverse de l'annulateur du A-module $A_S^{(n)}/A[X] \cap A_S^{(n)}$

COROLLAIRE 1 (Polya). *A_S est libre avec une base de polynômes de degré croissant si et seulement si tous les idéaux I_n sont principaux.*

C'est une conséquence immédiate du Théorème 2. On peut introduire la:

Définition. A_S est bien libre s'il admet une base de polynômes de degré croissant.

On peut ajouter à ce sujet, de façon evidente:

COROLLAIRE 1 (bis). *On a de façon équivalente:*
(i) *A_S est bien libre.*
(ii) *Tous les modules $A_S^{(n)}$ sont libres.*

Notons bien toutefois que le A-module A_S *est toujours libre* puisque projectif de rang infini [**1**; VII, § 4, Exercises].

COROLLAIRE 2. *Si A est l'anneau des entiers d'un corps de nombres K, $n!\, I_n$ est un idéal entier de A. Autrement dit, si P est dans A_S et de degré n, $n!P$ est dans $A[X]$.*

Notons v_p la valuation p-adique de $\mathbf{Q}$, v une valuation de K qui prolonge donc une des valuations v_p, et e l'indice de ramification de v_p sur v. Comme la norme N_v de v est une puissance de p (puissance égale au degré résiduel de v_p sur v) on a

$$\begin{aligned} S_v(n) &= \sum_{\alpha=1}^{\infty} \left[\frac{n}{N_v^{\alpha}}\right] \\ &\leqq \sum_{\alpha=1}^{\infty} \left[\frac{n}{p^{\alpha}}\right] \\ &= S_{v_p}(n) \\ &= \sum_{h=1}^{n} v_p(h) \\ &= \frac{1}{e} v(n!) \\ &\leqq v(n!). \end{aligned}$$

Comme $v(I_n^{-1}) = S_v(n)$ on a donc $v[n!I_n] \geqq 0$. Comme la suite $I_0, I_1, \ldots, I_n$ est croissante, les idéaux $n!I_0, n^1I_1, \ldots, n!I_n$ sont tous entiers, et si P est dans A_S et de degré n, $n!P$ est dans $A[X]$.

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INTEGRAL MEANS AND ZERO DISTRIBUTIONS OF BLASCHKE PRODUCTS

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1. Preliminaries. A sequence $\{z_n\}$ in $D = \{z: |z| < 1\}$ is a Blaschke sequence if and only if

$$\sum_n (1 - |z_n|) < \infty.$$

If 0 appears m times in $\{z_n\}$ then

$$B(z, \{z_n\}) = z^m \prod_{z_n \neq 0} \frac{\bar{z}_n(z_n - z)}{|z_n|(1 - z\bar{z}_n)}$$

is the Blaschke product defined by $\{z_n\}$. The set of all Blaschke products will be denoted by $\mathscr{B}$. If $B \in \mathscr{B}$ it is well-known that B is regular in D, and $|B(z, \{z_n\})| < 1$ when $z \in D$.

For a given pair of values p in $(0, \infty)$ and q in $[0, \infty)$ we denote by $\mathscr{I}(p, q)$ the class of all Blaschke products $B(z, \{z_n\})$ such that

$$I(r, p, \{z_n\}) = \int_0^{2\pi} |\log|B(re^{i\theta}, \{z_n\})||^p d\theta = O((1 - r)^{-q})$$

as $r \to 1 - 0$. In the case $q \geqq \max(p - 1, 0)$ the classes of functions $\mathscr{B}$ and $\mathscr{I}(p, q)$ are identical: this is a particular case of an elementary theorem for functions subharmonic in a disc, the analogous theorem for functions subharmonic in a half-plane appearing in [**1**].

If $0 \leqq q < \max(p - 1, 0)$ then $\mathscr{I}(p, q)$ is a proper subset of $\mathscr{B}$, and we examine the distributions of the zeros of Blaschke products in $\mathscr{I}(p, q)$ in these cases. The theorems obtained here are generalizations of results obtained elsewhere [**2**; **3**] in the case where $p = 2$ and $q = 0$. Although the earlier methods can be readily modified to prove the results of this paper for any non-negative q when $p = 2$, such modifications are not possible for other values of p except in the case of Theorem 3. Thus a different, and more direct, approach will be made to the proofs of our other theorems.

In stating our main results we use the notation

$$A(r, \theta, \psi) = \begin{cases} \{z : |\theta - \arg z| \leqq 2\psi, r < |z| \leqq \frac{1}{2}(1 + r)\}, & 0 \leqq \psi < \frac{1}{2}\pi, \\ \{z : r < |z| \leqq \frac{1}{2}(1 + r)\}, & \psi \geqq \frac{1}{2}\pi, \end{cases}$$

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and let $\nu(r, \theta, \psi, \{z_n\})$ denote the number of elements of $\{z_n\}$ in $A(r, \theta, \psi)$. Then we have the following theorem.

THEOREM 1. *Let* $p > 1$, $0 \leqq q < p - 1$, *and suppose that* $\{z_n\}$ *is a Blaschke sequence such that*

$$I(\rho, p, \{z_n\}) < C(1-\rho)^{-q},\ 0 < \rho < 1$$

for some constant C. *Then if* $\gamma \geqq 1$ *we have*

$$\nu(r, \varphi, (1-r)^\gamma, \{z_n\}) < K(p, q, \gamma)C^{1/p}(1-r)^{-(1+q)/p}, \quad 0 < r < 1,\ 0 \leqq \varphi < 2\pi, \tag{1.1}$$

and if $0 < \gamma < 1$ *we have*

$$\nu(r, \varphi, (1-r)^\gamma, \{z_n\}) < K(p, q, \gamma)C^{1/p}(1-r)^{\gamma-1-\gamma(1+q)/p}, \quad 0 < r < 1,\ 0 \leqq \varphi < 2\pi. \tag{1.2}$$

Here, and throughout the paper, we use the symbol K to denote a constant depending on the particular parameters under consideration. The values of K need not be the same in any two successive appearances, but at each appearance there will be some means of determining its value in terms of the relevant parameters, these being indicated either directly or by implication.

Next we show that the indices appearing in the right-hand sides of (1.1) and (1.2) are best possible. We do this by means of the following two theorems.

THEOREM 2. *If* $n(r, \{z_n\})$ *denotes the number of elements of* $\{z_n\}$ *for which* $|z_n| \leqq r$, *and*

$$n(r, \{z_n\}) < C(1-r)^{-\alpha},\ 0 < r < 1 \tag{1.3}$$

for some positive constant C *and some constant* α *in* $(0, 1)$, *then for each* p *in* $(1, \infty)$ *we have*

$$I(r, p, \{z_n\}) < K(\alpha, p)C^p(1-r)^{1-\alpha p},\ 0 < r < 1. \tag{1.4}$$

THEOREM 3. *Let*

$$z_n = (1 - n^{-\beta})e^{in^{-\beta\gamma}},\ n = 2, 3, \ldots,$$

where $0 < \gamma \leqq 1$, $\beta > 1$, *and* $p > 1$. *Then if*

$$\beta\gamma\lambda = p(\beta - 1) - \beta\gamma(p - 1) \tag{1.5}$$

we have

$$I(r, p, \{z_n\}) < K(p, \beta, \gamma, \lambda)((1-r)^{\lambda\gamma} + (1-r)^{\lambda} + (1-r)^{\gamma}), \quad 0 < r < 1, \tag{1.6}$$

while

$$\nu(r, 0, (1-r)^\gamma, \{z_n\}) \sim K(1-r)^{-1/\beta} \tag{1.7}$$

as $r \to 1 - 0$.

MacLane and Rubel [**2**] have proved Theorem 2 in the case where $p = 2$ and $\alpha = \frac{1}{2}$. As an immediate corollary of Theorem 1 and Theorem 2 we note the following generalization of another result proved by these two authors in the same paper.

COROLLARY. *If* $p > 1, 0 \leqq q < p - 1$, *and the elements of* $\{z_n\}$ *are contained in a finite number of Stolz angles, then* $B(z, \{z_n\}) \in \mathscr{I}(p, q)$ *if and only if*

$$n(r, \{z_n\}) = O((1-r)^{-(q+1)/p})$$

as $r \to 1 - 0$.

As well as showing that (1.1) is best possible, this Corollary demonstrates the existence of Blaschke products that do not belong to $\mathscr{I}(p, q)$ for any pair of numbers p and q for which $0 \leqq q < p - 1 < \infty$. For example

$$z_n = 1 - \frac{1}{n(\log n)^2}, n = 2, 3, 4, \ldots$$

defines a Blaschke product $\{z_n\}$ of this type.

In order to see that (1.2) is best possible, we put $\lambda = -q \leqq 0$ in the inequality (1.6) of Theorem 3. Then β can be chosen in the range $(1, \infty)$ to satisfy (1.5), so that

$$I(r, p, \{z_n\}) = O((1-r)^{-q})$$

as $r \to 1 - 0$ since $\lambda \leqq \lambda\gamma \leqq 0$. Further the relation (1.7) becomes

$$\nu(r, 0, (1-r)^\gamma, \{z_n\}) \sim K(1-r)^{\gamma-1-\gamma(1+q)/p}$$

as $r \to 1 - 0$, as required.

Finally we note that the proof of Theorem 3 has already been given elsewhere [**3**] in the case $p = 2$. The proof given in this special case needs only obvious amendments to apply in the generality stated here. Thus we may forego the details of the proof of Theorem 3, and we prove only Theorems 1 and 2 in this paper.

2. The proof of Theorem 1. Without loss of generality we suppose that $\varphi = 0$ and that $z_n = r_n e^{i\theta_n} \neq 0$ for all natural numbers n. Then it is easily shown (see e.g. [**2**]) that

$$\log|B(\rho e^{i\theta}, \{z_n\})|^{-2} = \sum_{n=1}^{\infty} \log\left(1 + \frac{(1-\rho^2)(1-r_n^2)}{P(\rho, r_n, \theta, \theta_n)}\right), \tag{2.1}$$

where

$$P(\rho, r_n, \theta, \theta_n) = (\rho - r_n)^2 + 4\rho r_n \sin^2 \tfrac{1}{2}(\theta - \theta_n).$$

For a given value r in $(\frac{3}{4}, 1)$ we suppose that

$$0 < \rho \leqq 4r - 3, \tag{2.2}$$

and

$$2(1-r)^\gamma < \theta < \pi.$$

Then, if $z_n \in A(r, 0, (1-r)^\gamma) = \mathscr{A}$, we have

$$\max\{3(1-r_n), \tfrac{3}{4}(1-\rho)\} < r_n - \rho < 1 - \rho, \tag{2.3}$$

$$\sin \tfrac{1}{2}(\theta - \theta_n) < 2 \sin \tfrac{1}{4}(\theta - \theta_n) < 2 \sin \tfrac{1}{2}\theta, \tag{2.4}$$

and it follows that

$$\begin{aligned} P(\rho, r_n, \theta, \theta_n) &< (1-\rho)^2 + 16 \sin^2 \tfrac{1}{2}\theta \\ &< (1 - \rho + 4 \sin \tfrac{1}{2}\theta)^2, \end{aligned} \tag{2.5}$$

while

$$P(\rho, r_n, \theta, \theta_n) > \frac{9}{4}(1-\rho)(1-r_n) > \frac{1}{2}(1-\rho^2)(1-r_n^2). \tag{2.6}$$

Now (2.6) implies that

$$\log\left(1 + \frac{(1-\rho^2)(1-r_n^2)}{P(\rho, r_n, \theta, \theta_n)}\right) > \frac{(1-\rho^2)(1-r_n^2)}{3P(\rho, r_n, \theta, \theta_n)},$$

so that, by (2.1) and (2.5), we have

$$\begin{aligned} \log|B(\rho e^{i\theta}, \{z_n\})|^{-2} &> \sum_{z_n \in \mathscr{A}} \frac{(1-\rho)(1-r_n)}{3P(\rho, r_n, \theta, \theta_n)} \\ &> \frac{\nu(r, 0, (1-r)^\gamma)(1-\rho)(1-r)}{6(1 - \rho + 4 \sin \frac{1}{2}\theta)^2}. \end{aligned}$$

Consequently, the hypothesis of Theorem 1 shows that

$$\begin{aligned} C(1-\rho)^{-q} &> I(\rho, p, \{z_n\}) \\ &> \frac{\nu(r, 0, (1-r)^\gamma)^p (1-\rho)^p (1-r)^p}{6^p} \int_{(1-r)^\gamma}^{\pi} \frac{d\theta}{(1 - \rho + 2\theta)^{2p}} \\ &> \frac{K\nu(r, 0, (1-r)^\gamma)^p (1-\rho)^p (1-r)^p}{(1 - \rho + 2(1-r)^\gamma)^{2p-1}}. \end{aligned}$$

In the case where $\gamma \geqq 1$ we put $\rho = 4r - 3$. Then

$$\nu(r, 0, (1-r)^\gamma)^p < KC(1-r)^{-1-q},$$

which gives (1.1). In the case where $0 < \gamma < 1$ we put $1 - \rho = (1-r)^\gamma$. Then the condition (2.2) is satisfied if $1 - r$ is sufficiently small, and we have

$$\nu(r, 0, (1-r)^\gamma)^p < KC(1-r)^{p(\gamma-1)-\gamma(q+1)},$$

which gives (1.2).

3. The proof of Theorem 2. In proving Theorem 2 we require the following elementary lemmas.

LEMMA 1. *If $a_j \geqq 0$ for $j = 1, 2, \ldots, N$, and $p > 1$, then*

$$\left[\sum_{j=1}^{N} a_j\right]^p \leqq N^{p-1} \sum_{j=1}^{N} a_j^p.$$

LEMMA 2. *If* $\{z_n\}$ *is a Blaschke sequence and* $0 < r < 1$ *then*

$$\sum_{|z_n|>r} (1 - |z_n|) \leqq \int_r^1 n(t, \{z_n\})dt.$$

LEMMA 3. *If* $a > 0$, $b > 0$, *and* $p > 1$ *then*

$$\int_0^{2\pi} \frac{d\theta}{(a^2 + b^2 \sin^2 \frac{1}{2}\theta)^p} < \frac{K(p)}{ba^{2p-1}}.$$

Without loss of generality we suppose $\frac{1}{2} < r < 1$ and $\frac{1}{2} < r_n < 1$, where again we denote z_n by $r_n e^{i\theta_n}$. Then the expression (2.1) gives immediately

$$\log|B(re^{i\theta}, \{z_n\})|^{-2} < \sum_{n=1}^{\infty} \frac{4(1 - r)(1 - r_n)}{(r - r_n)^2 + \sin^2 \frac{1}{2}(\theta - \theta_n)}. \tag{3.1}$$

Let $B_1(re^{i\theta})$, $B_2(re^{i\theta})$, and $B_3(re^{i\theta})$ be respectively the subproducts of $B(re^{i\theta}, \{z_n\})$ for which

$$r_n \leqq 2r - 1,\ 2r - 1 < r_n < \tfrac{1}{2}(1 + r), \text{ and } r_n \geqq \tfrac{1}{2}(1 + r).$$

The inequality (3.1) leads to

$$\begin{aligned}|\log|B_1(re^{i\theta})||^p &< K\left(\sum_{r_n \leq 2r-1} \frac{(1 - r)(1 - r_n)}{\{(r - r_n)^2 + \sin^2 \frac{1}{2}(\theta - \theta_n)\}}\right)^p \\ &< K(n(2r - 1))^{p-1} \sum_{r_n \leq 2r-1} \frac{(1 - r)^p(1 - r_n)^p}{\{(1 - r_n)^2 + \sin^2 \frac{1}{2}(\theta - \theta_n)\}^p},\end{aligned}$$

by application of Lemma 1 and the abbreviation $n(t) = n(t, \{z_n\})$. Hence, by Lemma 3, we have

$$\begin{aligned}\int_0^{2\pi} |\log|\, B_1(re^{i\theta})||^p d\theta &< K(n(2r - 1))^{p-1} \sum_{r_n \leq 2r-1} (1 - r)^p(1 - r_n)^{1-p} \\ &< K(1 - r)(n(2r - 1))^p,\end{aligned} \tag{3.2}$$

since $0 < 2(1 - r) \leqq 1 - r_n$.

A similar consideration of $B_2(re^{i\theta})$ leads to

$$\begin{aligned}\int_0^{2\pi} |\log|B_2(re^{i\theta})||^p d\theta &< Kn(\tfrac{1}{2}(1 + r))^{p-1} \\ &\times \sum_{2r-1<r_n<\frac{1}{2}(1+r)} \int_0^{2\pi} \left\{\log\left(1 + \frac{2(1 - r)^2}{\sin^2 \frac{1}{2}(\theta - \theta_n)}\right)\right\}^p d\theta \\ &< Kn(\tfrac{1}{2}(1 + r))^p(1 - r).\end{aligned} \tag{3.3}$$

Finally, we note that

$$\begin{aligned}\log|B_3(re^{i\theta})| &< K \sum_{r_n \geq \frac{1}{2}(1+r)} \frac{(1 - r_n)}{1 - r} \\ &\leqq \frac{K}{1 - r} \int_{\frac{1}{2}(1+r)}^1 n(t)dt \\ &< KC(1 - r)^{-\alpha}.\end{aligned}$$

Hence, since $p > 1$, we have

$$(3.4)\quad \int_0^{2\pi} |\log|B_3(re^{i\theta})||^p d\theta < KC^{p-1}(1-r)^{\alpha(1-p)} \int_0^{2\pi} |\log|B_3(re^{i\theta})||d\theta$$

$$< KC^{p-1}(1-r)^{\alpha(1-p)}$$

$$\times \sum_{r_n \geq \frac{1}{2}(1+r)} \int_0^{2\pi} \frac{(1-r)(1-r_n)}{(1-r)^2 + \sin^2\frac{1}{2}(\theta - \theta_n)} d\theta$$

$$< KC^p(1-r)^{1-\alpha p},$$

the change of order of the operations of integration and summation being possible by uniform convergence.

Substituting from (1.3) in (3.2) and (3.3), and noting that $B(z, \{z_n\}) = B_1(z)B_2(z)B_3(z)$, we obtain (1.4) by application of Minkowski's inequality.

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GENERALISATION DE LA FORMULE DE RIEMANN-HURWITZ

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Introduction: nombre de Chern. Etant un polynôme $\Phi \in R[X_1, \ldots, X_n]$ à n indéterminés, Φ est dit de poids homogène m si

$$\Phi(X_1, \ldots, X_n) = \sum q_{\alpha_1\alpha_2 \ldots \alpha_n} X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_n^{\alpha_n},$$

avec $m = \sum_{1 \leq p \leq n} 2p\alpha_p$.

Si E est un fibré vectoriel complexe de rang n (i.e., n est la dimension sur **C** de la fibre E_X) sur une variété différentiable orientée compacte M de dimension m, à tout polynôme Φ à n indéterminés de poids homogène m, on associe un nombre de Chern:

$$\begin{aligned}\Phi(E, M) &= <\Phi(E), [M]> \\ &= <\Phi(C_1(E), \ldots, C_n(E)), [M]>\end{aligned}$$

où $[M]$ est le cycle fondamental défini par l'orientation de M et $C_i(E)$ la ième classe de Chern de E. Dans le cas où le poids homogène de Φ est différent de la dimension de M, on pose: $\Phi(E, M) = 0$.

Il est utile de rappeler [**2**; **4**] que si ∇ est une connexion hermitienne sur E, i.e., ∇ est un opérateur différentiel linéaire

$$\nabla : \wedge^p T^*c \otimes E \rightarrow \wedge^{p+1} T^*c \otimes E,$$

T^*c étant le fibré des 1-formes complexes sur M, telle que

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \wedge \nabla(s),$$

pour toute p-forme complexe ω et section s de E sur M, on définit l'opérateur de courbure

$$\nabla^2 = \nabla \circ \nabla : E \rightarrow \Lambda^2 T^*c \otimes E$$

qui est un morphisme de fibré vectoriel. On peut considérer ∇^2 comme une section sur M de $\Lambda^2 T^*c \otimes \mathrm{End}(E)$, et définir

$$\det\left(\mathrm{Id} + \frac{\sqrt{-1}}{2\pi}\nabla^2\right) = 1 + C_1(\nabla) + \ldots + C_n(\nabla).$$

Alors, pour tout i, $C_i(\nabla)$ est une $2i$-forme réelle fermée dont la classe de cohomologie est par définition $C_i(E)$, la ième classe de Chern de E. Ainsi donc on a

$$\Phi(E, M) = \int_M \Phi(C_1(\nabla), \ldots, C_n(\nabla)).$$

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Formule de Riemann-Hurwitz généralisée. (a) Soient donnes deux fibrés vectoriels complexes E et F de rang n sur M. Soit Φ un morphisme de fibré vectoriel complexe sur M: $\Phi: E \to F$. Supposons qu'il existe une sous-variété différentiable fermée N de M telle que restreinte sur $M - N$, $\Phi: E|_{M-N}$ F_{M-N} soit un isomorphisme de fibrés vectoriels.

Considérons alors un voisinage tubulaire fermé $B(N)$ de N et son double différentiable

$$S(N) = B_1(N) \underset{\partial B(N)}{\cup} B_2(N)$$

obtenu de deux exemplaires distincts de $B(N)$ en identifiant leur bord. $S(N)$ est un fibré sur N en sphères S^r, r étant la codimension de N dans M. Il est immédiat de montrer que M étant compacte orientée, $S(N)$ est aussi une variété compacte orientable de même dimension m.

Sur $S(N)$, il existe alors un fibré vectoriel complexe canonique (E, ϕ, F) construit en prenant respectivement sur $B_1(N)$ et sur $B_2(N)$ la restriction de E et de F à $B(N)$ et en identifiant sur $B_1(N) \cap B_2(N) = \partial B(N)$ par l'isomorphisme de transition ϕ (dans la terminologie de [**1**], ϕ est ce qu'on appelle "clutching function").

Ceci étant, pour tout polynôme Φ de poids homogène m, on a

THÉORÈME 1 (Formule de Riemann-Hurwitz généralisée).

$$\Phi(E, M) - \Phi(F, M) = \Phi((E, \phi, F), S(N)),$$

le second membre étant défini par une orientation convenable de $S(N)$.

Preuve. En effet, soit donnée une connexion hermitienne ∇_1 sur E. Soit $B'(N)$ un voisinage tubulaire de N, contenu dans l'intérieur de $B(N)$. A l'aide d'une partition de l'unité on exhibe sur F une connexion hermitienne ∇_2 telle que sur $M - B'(N)$ on ait ce diagramme commutatif de faisceaux de sections:

$$\begin{tikzcd}
E \arrow[r, "\nabla_1"] \arrow[d, "\phi"'] & T^*\mathrm{c} \otimes E \arrow[d, "\mathrm{Id} \otimes \phi"] \\
F \arrow[r, "\nabla_2"] & T^*\mathrm{c} \otimes F.
\end{tikzcd}$$

D'où:

$$C_t(\nabla_1)|_{M-B(N)} = C_t(\nabla_2)|_{M-B(N)}.$$

Ainsi donc on a

$$(1)\quad \phi(E, M) - \phi(F, M) = \int_M \phi(C_1(\nabla_1), \dots, C_n(\nabla_1)) - \phi(C_1(\nabla_2), \dots, C_n(\nabla_2))$$
$$= \int_{B(N)} \Phi(C_1(\nabla_1), \dots, C_n(\nabla_1)) - \Phi(C_1(\nabla_2), \dots, C_n(\nabla_2)).$$

Or, par la méthode standard de raccordement, on montre facilement qu'il existe une connexion hermitienne ∇ sur (E, ϕ, F) telle que sur

$$(E, \phi, F)|_{B_1(N)} \simeq E|_{B(N)}, \ \nabla \text{ soit égale à } \nabla_1,$$

et sur

$$(E, \phi, F)|_{B_2(N)} \simeq F|_{B(N)}, \ \nabla \text{ soit égale à } \nabla_2.$$

D'où le second membre de (1) n'est autre que

$$\int_{S(N)} \Phi(C_1(\nabla), \ldots, C_n(\nabla)) = \Phi((E, \phi, F), S(N))$$

pour une orientation convenable de $S(N)$.

(b) Supposons que sur N, ϕ est de rang constant, ce qui entraîne que sur N, nous avons une suite exacte de fibrés vectoriels.

$$0 \to K_1 \to E|_N \xrightarrow{\phi} F|_N \to K_2 \to 0.$$

Nous désignerons par L, le sous-fibré de $F|_N$, qui est l'image $\phi(E|_N)$. Ainsi donc, on a ces isomorphismes

$$\begin{aligned} E|_N &\simeq K_1 \oplus L, \\ F|_N &\simeq L \oplus K_2. \end{aligned}$$

Désignons par $\pi : B(N) \to N$, la fibration de $B(N)$ sur N. Comme N est un rétract par déformation de $B(N)$, on a:

$$\begin{aligned} E|_{B(N)} &\simeq \pi^*K_1 \oplus \pi^*L, \\ F|_{B(N)} &\simeq \pi^*K_2 \oplus \pi^*L, \end{aligned} \tag{2}$$

où π^*L, π^*K_1, π^*K_2 sont respectivement des fibrés induits par π sur $B(N)$. L'image par ϕ: $E|_{B(N)} \to F|_{B(N)}$ du sous-fibré π^*L de $E|_{B(N)}$ est un sous-fibré isomorphe à π^*L de $F|_{B(N)}$ et dont le fibré quotient est isomorphe à π^*K_2. Donc on peut supposer que modulo des isomorphismes de (2), l'isomorphisme ϕ soit de la forme

$$\begin{tikzcd} \phi : E|_{\partial B(N)} \arrow[r] \arrow[d, phantom, "\wr"] & F|_{\partial B(N)} \arrow[d, phantom, "\wr"] \\ \pi^*K_1 \oplus \pi^*L \arrow[r] & \pi^*K_2 \oplus \pi^*L \\ (x, y) \arrow[r] & (\psi(x), y) \end{tikzcd}$$

où ψ est un isomorphisme de fibrés vectoriels:

$$\pi^*K_1|_{\partial B(N)} \to \pi^*K_2|_{\partial B(N)}.$$

Autrement dit, dans le cas considéré, on a le lemme suivant:

Lemme. *Sur $S(N)$, on a l'isomorphisme de fibrés vectoriels*

$$(E, \phi, F) \simeq \pi^*L \oplus (\pi^*K_1, \psi, \pi^*K_2)$$

où le second membre est la somme directe de π^*L, le fibré induit de L sur $S(N)$ par la fibration $\pi\colon S(N) \to N$ et de $(\pi^*K_1, \psi, \pi^*K_2)$, qui est le fibré construit sur

$$S(N) = B_1(N) \bigcup_{\partial B(N)} B_2(N)$$

par une certaine fonction de transition ψ.

Nous désignerons, pour simplifier, par $1 + C_1(K) + \ldots + C_p(K)$ la classe totale de Chern sur $S(N)$ de $(\pi^*K_1, \psi, \pi^*K_2)$. Et si $1 + C_1(L) + \ldots + C_{n-p}(L)$ est la classe totale de Chern de L sur N, $\pi^*(1 + C_1(L) + \ldots + C_{n-1}(L))$, la classe induite par $\pi\colon S(N) \to N$, sera celle de π^*L sur $S(N)$.

Supposons dans la suite que N est une sous-variété orientable connexe de M. On a la proposition suivante:

PROPOSITION. *Soient Φ_1 et Φ_2 des polynômes respectivement de p et de $n-p$ indéterminés et de poids homogènes r et $n-r$ ($r =$ codim N). On a*

$$\langle \Phi_1(C_1(K), \ldots, C_p(Kg) \wedge \pi^*\Phi_2(C_1(L), \ldots, C_{n-p}(L)); [S(N)]\rangle = k\Phi_2(L, N)$$

ou k est une constante et $\Phi_2(L, N)$ est le nombre de Chern défini par Φ_2 relativement a une certaine orientation de N.

Preuve. Il suffit de se donner des connexions hermitiennes ∇ sur $(\pi^*K_1, \psi, \pi^*K_2)$ et ∇_1 sur L. On a

$$\begin{aligned}\langle \Phi_1(C_1(K), \ldots, C_p(K)) \wedge \pi^*\Phi_2(C_1(L), \ldots, C_{n-p}(L)), [S(N)]\rangle \\ = \int_{S(N)} \Phi_1(C_1(\nabla), \ldots, C_p(\nabla)) \wedge \pi^*\Phi_2(C_1(\nabla_1), \ldots, C_{n-p}(\nabla_1)).\end{aligned}$$

Par intégration par partie le long des fibres $S(N)|_{x \in N}$ de $S(N)$ sur N, le second membre est aussi

$$\int_N \Big(\int_{S(N)|_{x\in N}} \Phi_1(C_1(\nabla), \ldots, C_p(\nabla))\Big) \wedge \Phi_2(C_1(\nabla_1), \ldots, C_{n-p}(\nabla_1)).$$

Or

$$\int_{S(N)|_{x\in N}} \Phi_1(C_1(\nabla), \ldots, C_p(\nabla)) = \Phi_1(K_x, S(N)|_x)$$

où

$$K_x = (\pi^*K_1|_{B(N)_x}, \psi_x, \pi^*K_2|_{B(N)_x})$$

est le fibré construit sur $S(N)|_x$ à l'aide de la fonction de transition ψ_x,

$$\psi_x\colon \pi^*K_1|_{\partial B(N)_x} \xrightarrow{\sim} \pi^*K_2|_{\partial B(N)_x},$$

la restriction de ψ à $\partial B(N)_x$.

Or si $\gamma(x, y)$ est une courbe dans N, joignant deux points x et y, $K_1|_{\gamma(x,y)}$ et $K_2|_{\gamma(x,y)}$ sont triviaux. Donc, on a des isomorphismes verticaux dans le diagramme suivant:

$$\begin{tikzcd}
\pi^*K_1|_{\partial B(N)_x} \arrow[r, "\psi_x"] \arrow[d, phantom, "\wr"] & \pi^*K_2|_{\partial B(N)_x} \arrow[d, phantom, "\wr"] \\
\pi^*K_1|_{\partial B(N)_y} \arrow[r, "\psi_y"] & \pi^*K_2|_{\partial B(N)_y},
\end{tikzcd}$$

et modulo ces isomorphismes, ψ_x et ψ_y sont homotopes. Ainsi K_x et K_y, considérés comme fibrés sur la sphère

$$S^r \simeq S(N)|_x \simeq S(N)|_y,$$

sont isomorphes; ce qui implique

$$\Phi_1(K_x, S(N)|_x) = k,$$

constante indépendante de x, la variété N étant connexe par arc. D'où

$$\int_N \Big(\int_{S(N)|_x} \Phi_1(C_1(\nabla), \dots, C_p(\nabla))\Big) \wedge \Phi_2(C_1(\nabla_1), \dots, C_{n-p}(\nabla_1))$$
$$= k\int_N \Phi_2(C_1(\nabla_1), \dots, C_{n-p}(\nabla_1)) = k\Phi_2(L, N).$$

(c) Un cas important des considérations précédentes est lorsque $\Phi \equiv X_n \in R[X_1, \dots, X_n]$. Alors, pour tout fibré complexe E de rang n sur M, on a $\Phi(E) = C_n(E)$, la classe de Chern de plus grand degré de E. D'après le théorème de Gauss-Bonnet [**4**], $C_n(E)$ n'est autre que la classe d'Euler-Poincaré de E.

Si la dimension de M est deux fois le rang du fibré complexe E, nous noterons

$$\langle C_n(E), [M]\rangle = \chi(E, M).$$

Ceci dit, d'après la formule de Whitney, on a, pour

$$(E, \Phi, F) = \pi^*L \oplus (\pi^*K_1, \psi, \pi^*K_2)$$

dans les hypothèses du lemme de la section (b),

$$C_n(E, \phi, F) = \pi^*C_{n-p}(L) \wedge C_p(K).$$

Immédiatement par application de la proposition de la section (b), on a

THÉORÈME 2 (Formule de Riemann-Hurwitz). *Sous les hypothèses de la section* (b) *précédente, on a*:

$$\chi(E, M) - \chi(F, M) = k\chi(L, N)$$

le rang $(n - p)$ *du fibré complexe* L *étant égal a la moitié de la dimension de* N.

La formule est encore évidemment valable avec le second membre nul lorsque $2(n - p) > m - r$, dimension de N.

(d) *Cas de fibrés réels.* Nous nous sommes restreints dans tout ce qui précède à la catégorie des fibrés vectoriels complexes. Mais si E et F sont des fibrés réels sur M, le morphisme ϕ étant alors un morphisme de fibrés réels, avec des hypothèses identiques, on a de même la formule de Riemann-Hurwitz généralisée en considérant non plus des nombres de Chern mais des nombres de Pontrjagin respectivement de E et F sur M et de (E, ϕ, F) sur $S(N)$.

En particulier, soient E et F des fibrés vectoriels réels orientés sur M. Dans les hypothèses de la section (b), supposons en plus que L, étant l'image de

$$\phi\colon E|_N \to F|_N,$$

soit un fibré vectoriel sur M orientable. Alors, on a de même la formule de Riemann-Hurwitz avec les nombres d'Euler:

$$\chi(E, M) - \chi(F, M) = k\chi(L, N)$$

le rang de L étant égal à la dimension de N. La formule reste encore valable avec second membre nul si le rang de L est plus grand que la dimension de N.

Application. Rappelons l'application bien connue de la formule de Riemann-Hurwitz aux revêtements ramifiés.

Soit $\pi: M \rightarrow S$ une application différentiable surjective d'une variété compacte orientée M sur une variété différentiable orientée de même dimension m. L'application π est un revêtement ramifié, s'il existe une sous-variété orientable N fermée de S, de codimension plus grande ou égale à 2, telle que:

(1) $\pi: M - \pi^{-1}(N) \rightarrow S - N$ soit une submersion conservant l'orientation et, $M - \pi^{-1}(N)$ soit aussi un revêtement connexe à q feuillets de $S - N$;

(2) $\pi^{-1}(N)$ soit une sous-variété fermée de M et $\pi: \pi^{-1}(N) \rightarrow N$ soit un difféomorphisme.

Remarquons alors que si N_0 est une composante connexe de N, il existe un voisinage tubulaire $B(N_0)$ tel que $\pi^{-1}(B(N_0))$ soit un voisinage tubulaire de $\pi^{-1}(N_0)$. Ainsi

$$\pi: \pi^{-1}(B(N_0)) - \pi^{-1}(N_0) \rightarrow B(N_0) - N_0$$

est un revêtement connexe à q feuillets de $B(N_0) - N_0$. Or, si N_0 est de codimension plus grande que 2, $B(N_0) - N_0$ est simplement connexe. Ce qui est impossible si $q > 1$. Nous pouvons donc supposer que N soit de codimension 2.

Ceci dit, considérons le morphisme tangent $d\pi$ du fibré vectoriel sur M.

$$\begin{tikzcd}
T(M) \arrow[rr, "d\pi"] \arrow[dr] & & \pi^*T(S) \arrow[dl] \\
& M &
\end{tikzcd}$$

Nous sommes dans les hypothèses de la formule de Riemann-Hurwitz: d'où

$$\chi(T(M), M) - \chi(\pi^*(T(S)), M) = k\chi(T(\pi^{-1}(N)), \pi^{-1}(N))$$

où encore

$$\chi(M) - q\chi(S) = k\chi(N).$$

On démontre facilement que k est égal au signe près à $q - 1$: De fait, on a:

$$\chi(M) - q\chi(S) = -(q - 1)\chi(N).$$

En effet k étant égal au signe près à $(q - 1)$, pour déterminer le signe, comme la formule est évidemment universelle, il suffit de regarder, sur un exemple précis. Or, on peut voir facilement qu'il existe un revêtement ramifié à deux feuillets de la sphère S^2 par un tore $S^1 \times S^1$ avec quatre points de ramifications. D'où:

$$\begin{aligned} \chi(S^1 \times S^1) - 2\chi(S^2) &= 4k \\ 0 - 4 &= 4k \\ k &= -2 = -(q-1). \end{aligned}$$

Exemple (qui m' a été suggéré par J. Morrow, Washington University, Seattle). Soit D une courbe algébrique non singulière de degré q de l'espace projectif $P^2(\mathbf{C})$. Rappelons que d'après Kodaira [**3**, p. 119]

$$\chi(D) = +(-q^2 + 3q).$$

D'autre part le diviseur $[D]$ définit évidemment un fibré vectoriel complexe sur $P^2(\mathbf{C})$ isomorphe à $H \otimes \ldots \otimes H = H^q$, H étant le fibré universel de Hopf.

Soit un recouvrement de $P^2(\mathbf{C})$ par des ouverts U_i de coordonnées.

Désignons par $\phi_i = 0$, l'équation de définition de $D \cap U_i$ dans U_i.

Si ξ_i est la coordonnée vectorielle de $H|_{U_i}$,

$$\begin{aligned} H|_{U_i} &\simeq U_i \times \mathbf{C} \\ x &\to (\pi(x), \xi_i(x)) \end{aligned}$$

où π: $H \to P^2(\mathbf{C})$ est l'application de fibration, alors dans $H|_{U_i}$, l'équation $\phi_i \circ \pi = \xi_i{}^q$ définit une sous-variété $V_i \subseteq H|_{U_i}$. Il n'est pas difficile de montrer que V_i se raccordent pour définir une sous-variété non singulière V de H et que V est revêtement ramifié à q feuillets de $P^2(_U)$, ramifié le long de la sous-variété D de $P^2(\mathbf{C})$. Ainsi on a:

$$\begin{aligned} \chi(V) - q\chi(P^2(\mathbf{C})) &= -(q-1)\chi(D) \\ \chi(V) - 3q &= -(q-1)(3q - q^2) \\ \chi(V) &= q(6 - 4q + q^2). \end{aligned}$$

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TAMING WILD SIMPLE CLOSED CURVES WITH MONOTONE MAPS

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1. Introduction. Hempel [**6**, Theorem 2] proved that if S is a tame 2-sphere in E^3 and f is a map of E^3 onto itself such that $f|S$ is a homeomorphism and $f(E^3 - S) = E^3 - f(S)$, then $f(S)$ is tame. Boyd [**4**] has shown that the converse is false; in fact, if S is any 2-sphere in E^3, then there is a monotone map f of E^3 onto itself such that $f|S$ is a homeomorphism, $f(E^3 - S) = E^3 - f(S)$, and $f(S)$ is tame.

It is the purpose of this paper to prove that the corresponding converse for simple closed curves in E^3 is also false. We show in Theorem 4 that if J is any simple closed curve in a closed orientable 3-manifold M^3, then there is a monotone map $f: M^3 \to S^3$ such that $f|J$ is a homeomorphism, $f(J)$ is tame and unknotted, and $f(M^3 - J) = S^3 - f(J)$.

In Theorem 1 of § 2, we construct a cube-with-handles neighbourhood of a simple closed curve in an orientable 3-manifold. This neighbourhood is a solid torus, sectioned into 3-cells, with a small cube-with-handles attached to each section to cover a small subarc of J associated with that section.

Theorem 1′ constructs an analogous neighbourhood for finite graphs.

In § 3 we extend the construction given in § 2 to give a cube-with-handles neighbourhood of a simple closed curve in which the simple closed curve is homotopic to a simple closed curve lying in the boundary of the solid torus portion of the neighbourhood. Similar extensions are given for neighbourhoods of finite graphs.

Sections 4 and 5 construct an infinite sequence of cube-with-handles neighbourhoods similar to those of Theorem 1, each lying "nicely" in the previous one. In the process of constructing these neighbourhoods, it is shown that if J is homologous to zero, then J bounds an open surface.

In § 6, the infinite sequence of neighbourhoods is used to construct the monotone map of the 3-manifold onto S^3 which carries a simple closed curve in the manifold onto a tame unknotted simple closed curve. In the case that the simple closed curve J has a solid torus neighbourhood in which it is homologous to a centreline, there is a monotone map of the manifold onto itself which tames J and which is the identity outside the solid torus neighbourhood.

In § 7, we show that any knot, link, or wedge of simple closed curves in an orientable 3-manifold which is homologous to zero (respectively, contractible to a point) in the 3-manifold, is homologous to zero (respectively, contractible to a point) in a cube-with-handles in the 3-manifold.

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By a *map* or *mapping* we will mean a continuous function. If each point inverse of a map is compact and connected, then the map will be called *monotone*.

A *surface* is a 2-manifold. An *open manifold* is a noncompact manifold without boundary, and a *closed manifold* is a compact manifold without boundary. We will denote the boundary of a manifold M by ∂M, and the interior of M by $\operatorname{Int} M$. A surface S will be said to be *properly embedded* in a 3-manifold M if $\partial S \subset \partial M$ and $\operatorname{Int} S \subset \operatorname{Int} M$. We will assume that any manifold has a given metric, and we will denote this metric by the symbol ρ. The diameter of a set X will be denoted by $\operatorname{diam}(X)$.

A *punctured disk* is a disk D minus the interior of the union of a finite mutually disjoint collection of subdisks of the interior of D.

By a *graph*, we will mean a finite connected 1-complex. A vertex or 1-simplex v of a graph G has order n if v is a face of n 1-simplexes of G. The *star* of a vertex v of G is the closure of the union of the simplexes of G which have v as a face. An *n-frame* is the union of n arcs all intersecting at a common end point.

Let S be a 2-sided polyhedral surface in a 3-manifold M^3, and let A be an oriented polyhedral arc or simple closed curve which pierces S at each of its points of intersection with S. If A pierces S n more times in one direction than in the other, we call n the (unsigned) *algebraic intersection number* of A and S.

We use the fact that a polygonal simple closed curve in a 3-manifold M^3, which is homologous to zero in M^3, bounds a polyhedral orientable surface in M^3. Also, if two disjoint polygonal simple closed curves are homologous in M^3, then they bound a polyhedral orientable surface in M^3.

2. Neighbourhoods of finite graphs. In this section we construct neighbourhoods of finite graphs topologically embedded in a 3-manifold which are as close as we can make them to a regular neighbourhood. This neighbourhood is in fact the regular neighbourhood of a polygonal approximation to the graph with small cubes-with-handles attached along disks in the boundary of this regular neighbourhood to give a cube-with-handles neighbourhood of the topologically embedded graph. Near points where the topologically embedded graph is tame we do not need to attach the small cubes-with-handles. If the graph is polygonal our neighbourhood is, in fact, a regular neighbourhood of the graph. This neighbourhood will be used in §§ 4 and 5 to construct an infinite sequence of neighbourhoods of a simple closed curve which will in turn be applied in § 6 to define a monotone mapping carrying the simple closed curve to a tame unknotted simple closed curve in S^3.

THEOREM 1. *Let J be a simple closed curve topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, J has a cube-with-handles neighbourhood N with the following structure:*

(1) *There is a solid torus T with n meridional spanning disks $D_1, D_2, \ldots, D_n$ which divide T into n 3-cells $T_1, T_2, \ldots, T_n$ such that $D_i = T_i \cap T_{i+1}$ and $D_n = T_n \cap T_1$.*

(2) *There are n points $p_1, p_2, \ldots, p_n$ on J which divide J into n closed subarcs $J_1, J_2, \ldots, J_n$ such that $p_i = J_i \cap J_{i+1}$ and $p_n = J_n \cap J_1$.*
(3) $p_i \in \operatorname{Int} D_i$, *for each i.*
(4) *Each 3-cell T_i has an associated cube-with-handles H_i such that $T \cap H_i = T_i \cap H_i = (\partial T_i - D_i - D_{i-1}) \cap \partial H_i$ is a disk F_i.*
(5) $J_i \subset T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}$.
(6) $\operatorname{diam}(T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}) < \epsilon$.
(7) $N = T \cup (\bigcup H_i)$.
(8) *If J is locally tame at each point J_i, then $T_i \cap J = J_i$ is an unknotted spanning arc of T_i (hence there is no need for H_i).*

Remark. If M^3 is non-orientable, the same theorem is true except that T may be a solid Klein bottle, so N is a cube with (possibly) non-orientable handles.

Proof of Theorem 1. Let $\delta < \epsilon/25$. Choose points $p_1, p_2, \ldots, p_n$ of J dividing J into subarcs $J_1, J_2, \ldots, J_n$ of diameter less than $\delta/3$ such that

$$p_i = J_i \cap J_{i+1}, \quad i = 1, \ldots, n$$

(subscripts are understood to be integers mod n), and $J_i \cap J_j = \emptyset$ if $j \neq i-1$, i, or $i+1$. The arcs $J_1, J_2, \ldots, J_n$ form the 2-skeleton of a curvilinear triangulation of J with vertices $p_1, p_2, \ldots, p_n$.

Let J' be a polygonal approximation to J, where $J' = J_1' \cup J_2' \cup \ldots \cup J_n'$ is a simple closed curve with J_i' $\delta/3$-homotopic to J_i by a homotopy keeping the endpoints of J_i fixed. By [**9**, Lemma 3], J_i' can be adjusted slightly near $J_i' \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1}))$ so that J_i' is disjoint from

$$\operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})).$$

Thus we will assume that J' has this property for each subarc J_i'.

Take a polygonal solid torus neighbourhood T of J' and a disjoint collection of meridional disks $D_1, D_2, \ldots, D_n$ such that $D_i \cap J' = \{p_i\}$, $i = 1, 2, \ldots, n$, and $p_i \in \operatorname{Int} D_i$. If T_i is the closure of the component of $T - \bigcup\{D_i : i = 1, 2, \ldots, n\}$ containing $D_{i-1} \cup D_i$, then T_i is a 3-cell for each $i = 1, 2, \ldots, n$. The D_i's and T may be chosen so that $\operatorname{diam}(T_i) < \delta/3$, and because

$$J_i' \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})) = \emptyset,$$

we may assume that T and the D_i's were chosen so that

$$T_i \cap \operatorname{Cl}(J - (J_{i-1} \cup J_i \cup J_{i+1})) = \emptyset, \quad i = 1, 2, \ldots, n.$$

The latter condition insures that $J_i \cap \operatorname{Cl}(T - (T_{i-1} \cup T_i \cup T_{i+1})) = \emptyset$ for each i.

Consider the collection of sets $J_1 \cap \partial T, J_2 \cap \partial T, \ldots, J_n \cap \partial T$. This is a collection of mutually exclusive compact subsets of ∂T such that no component of any $J_i \cap \partial T$ separates a neighbourhood of itself in ∂T. Hence, there is a collection

$$\mathscr{D} = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \ldots \cup \mathscr{D}_n$$

of mutually exclusive disks in ∂T such that $\mathscr{D}_i$ is a mutually exclusive collection of disks containing $J_i \cap \partial T$ in the union of their interiors and $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ if $i \neq j$. Furthermore, $\mathscr{D}_i$ can be chosen so that the union $\mathscr{D}_i^*$ of the disks in $\mathscr{D}_i$ lies in $\partial(T_{i-1} \cup T_i \cup T_{i+1})$ missing the two end disks D_{i-2} and D_{i+1}, and since $J_i \cap \partial T$ is a $\delta/3$-set we may assume that each disk of $\mathscr{D}_i$ has diameter less than $\delta/3$.

By "sliding" each ∂D_i along ∂T we may adjust $\bigcup D_i$ so that $(\bigcup D_i) \cap \mathscr{D}^* = \emptyset$, no point of D_i is moved more than $\delta/3$, and $\mathscr{D}_i^*$ lies in $T_{i-1} \cup T_i \cup T_{i+1}$. We do this adjustment so close to each component of $\mathscr{D}$ that p_i is still in the adjusted D_i and the resulting T_i's retain the property that J_i does not meet any T_j unless $j = i - 1$, i, or $i + 1$, and $\operatorname{diam}(T_i) < \delta$.

As in [7, Lemma 2], let $\mathscr{D}^*_{i,j}$ denote the set $\mathscr{D}_i^* \cap \partial T_j$ for $j = i - 1$, i, $i + 1$. There are three mutually disjoint disks on $\partial T_j - D_{j-1} - D_j$, namely $B_{j-1,j}$, $B_{j,j}$, $B_{j+1,j}$ so that $\mathscr{D}^*_{j-1,j} \subset \operatorname{Int} B_{j-1,j}$, $\mathscr{D}^*_{j,j} \subset \operatorname{Int} B_{j,j}$ and $\mathscr{D}^*_{j+1,j} \subset \operatorname{Int} B_{j+1,j}$. Thus $\mathscr{D}_i^* \subset B_{i,i-1} \cup B_{i,i} \cup B_{i,i+1}$ and $B_{i,i-1} \subset \partial T_{i-1}$, $B_{i,i} \subset \partial T_i$, $B_{i,i+1} \subset \partial T_{i+1}$. There are two arcs, one joining $B_{i,i-1}$ to $B_{i,i}$ intersecting D_{i-1} precisely once, and one joining $B_{i,i}$ to $B_{i,i+1}$ intersecting D_i precisely once; both arcs are disjoint from any other $B_{j,k}$'s and lie in $\partial(T_{i-1} \cup T_i)$ and $\partial(T_i \cup T_{i+1})$, respectively.

It is easy to see that there is a disjoint collection of such arcs in ∂T such that each arc intersects $\bigcup \partial D_i$ precisely once and joins some $B_{i,i}$ to $B_{i,i-1}$ or some $B_{i,i}$ to $B_{i,i+1}$ and each $B_{i,i}$ is joined to $B_{i,i-1}$ by one such arc and to $B_{i,i+1}$ by one. Replacing these arcs by thin disks we obtain disks $F_1, F_2, \ldots, F_n$ on ∂T such that

$$\mathscr{D}_i^* \subset B_{i,i-1} \cup B_{i,i} \cup B_{i,i+1} \subset F_i,$$

$$F_i \subset \partial(T_{i-1} \cup T_i \cup T_{i+1}) - D_{i-2} - D_{i+1}, \quad \text{and} \quad F_i \cap F_j = \emptyset$$

if $i \neq j$. We now adjust the disks $D_1, D_2, \ldots, D_n$ near ∂T to slip them off $\bigcup F_i$ so that $F_i \subset \partial T_i - D_{i-1} - D_i$. This adjusts $T_1, T_2, \ldots, T_n$, also. We now have the structure of (1), (2), and (3) of the conclusions to the theorem. Since

$$\operatorname{diam}(F_i) \leqq \operatorname{diam}(T_{i-1} \cup T_i \cup T_{i+1}) < 3\delta$$

before this last adjustment, then

$$\operatorname{diam} T_i < \delta + 2(3\delta) = 7\delta$$

after pushing the D_i's off the F_i's.

Let M_i' be a compact 3-manifold with connected boundary intersecting T in a collection of punctured disks in the boundary of each of M_i' and T, with $M_i' \cap T \subset F_i$, $J_i - T \subset \operatorname{Int} M_i'$, $\operatorname{diam}(M_i') < \delta/3$, and $M_i' \cap M_j' = \emptyset$ if $i \neq j$. Fatten the disk F_i slightly into the complement of T and add the resulting cell to M_i' to obtain a compact 3-manifold with connected boundary M_i intersecting T in exactly the disk F_i.

There is a collection $\mathscr{A}_i$ of arcs in M_i such that each arc lies in Int M_i except that its endpoints lie in $\partial M_i - F_i$ and such that M_i minus a small tubular neighbourhood of every arc of $\mathscr{A}_i$ is a cube-with-handles. Such arcs exist by [**9**, Lemma 1]. By [**9**, Lemma 3], the collection of arcs $\mathscr{A}_i$ may be adjusted near $\mathscr{A}_i^* \cap J_i$ so that $J_i \cap \mathscr{A}_i^* = \emptyset$. Let H_i be the cube-with-handles obtained by removing small tubular neighbourhoods of these adjusted arcs of $\mathscr{A}_i$ from M_i. Then $H_i \cap T = F_i$ and

$$\begin{aligned}\operatorname{diam}(H_i) &\leqq \operatorname{diam}(M_i') + \operatorname{diam}(F_i)\\ &< \delta/3 + 3\delta = 3\tfrac{1}{3}\delta.\end{aligned}$$

The cube-with-handles H_i is the one promised in (4); and (5) follows. We let $N = T \cup (\bigcup H_i)$ and note that

$$\begin{aligned}\operatorname{diam}(T_{i-1} \cup (T_i \cup H_i) \cup T_{i+1}) &\leqq \operatorname{diam} T_{i-1} + \operatorname{diam} T_i + \operatorname{diam} H_i\\ &\qquad + \operatorname{diam} T_{i+1}\\ &< 7\delta + 7\delta + 3\tfrac{1}{3}\delta + 7\delta\\ &= 24\tfrac{1}{3}\delta < \epsilon.\end{aligned}$$

To obtain (8) we assume without loss of generality by [**3**, Theorem 9] that J is locally polyhedral mod its set of wild points. If J is locally tame at each point of J_i, then J_i is polyhedral and we can choose $J_i' = J_i$. It then follows that T, T_i, D_i and D_{i-1} can be so chosen as in (8). This completes the proof of Theorem 1.

Remark. Let p be a point of a (possibly wild) simple closed curve J and let U be a neighbourhood of p. Then there is a disk D in U such that $\partial D \cap J = \emptyset$ and any polygonal approximation of J which is homotopic to J in the complement of ∂D intersects D algebraically once. Just choose D to be a D_i of a sufficiently close neighbourhood N of J as constructed in Theorem 1.

A *special decomposition* P of a graph G is a decomposition of G into vertices, 1-simplexes, and n-frames obtained as follows from a triangulation of the graph which is so fine that the star of two vertices of order greater than 2 do not intersect: At each vertex v of order $n > 2$, replace v and each 1-simplex containing v with the n-frame star of v. The 1-simplexes and n-frames of the decomposition will be called 1-*elements*. The special decomposition P' of the graph G is a *subdivision* of P if each vertex of P is also a vertex of P'.

THEOREM 1′. *Let G be a finite graph topologically embedded in an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood N with the following structure:*

(1) *There is a special decomposition P of G and a cube-with-handles $T = \bigcup \{T_\sigma : \sigma$ is a 1-element of $P\}$, where each T_σ is a 3-cell associated with σ.*

(2) *$T_\sigma \cap T_{\sigma'} = \emptyset$ if $\sigma \cap \sigma' = \emptyset$ and $T_\sigma \cap T_{\sigma'} = D_\tau$, where D_τ is a disk in the boundary of each of T_σ and $T_{\sigma'}$ if $\tau = \sigma \cap \sigma'$ is a vertex P. In this case, $\tau \in \operatorname{Int} D_\tau$.*

(3) *Each 3-cell T_σ has an associated cube-with-handles H_σ such that $T \cap H_\sigma = T_\sigma \cap H_\sigma = (\partial T_\sigma - \bigcup \{D_\tau : \tau$ is a vertex of $\sigma\}) \cap \partial H_\sigma$ is a disk F_σ.*

(4) *If σ is a 1-element of P, then $\sigma \subset T_\sigma \cup H_\sigma \cup (\bigcup \{T_{\sigma'} : \sigma'$ is a 1-element of P and $\sigma' \cap \sigma \neq \emptyset\})$.*

(5) *If σ is a 1-element of P, then $\operatorname{diam}(\bigcup \{T_{\sigma'} \cup H_{\sigma'} : \sigma'$ is a 1-element of P and $\sigma' \cap \sigma \neq \emptyset\}) < \epsilon$.*

(6) *$N = T \cup (\bigcup H_\sigma)$.*

(7) *If G is locally tame at each point of the 1-element σ, then $T_\sigma \cap G = \sigma$ and σ lies in T_σ as the cone from an interior point of the 3-cell T_σ to a finite collection of points of ∂T_σ. In this case, there is no H_σ.*

Remark. If M^3 is non-orientable, T (and hence N) may be a cube with non-orientable handles; with this exception Theorem 1′ holds for a non-orientable M^3.

Proof of Theorem 1′. The proof is essentially the same as that of Theorem 1, except at the vertices of G of order $r > 2$. We indicate here how to modify the proof of Theorem 1. We take first of all a special decomposition P of the graph G instead of the triangulation of J. We choose a polygonal approximation G' of G $\delta/3$-homotopic to G keeping the vertices of P fixed; in particular, each vertex of G is also a vertex of G'. Instead of a solid torus neighbourhood of J', as in Theorem 1, we choose a regular neighbourhood T of G' and a collection of spanning disks D_τ of T, one for each vertex τ of P, which divides T into 3-cells satisfying (1) and (2). Note that there is a 3-cell T_σ for each 1-element σ of P and each T_σ is separated from "adjacent" $T_{\sigma'}$'s by a disk D_τ. If σ is an n-frame, note that T_σ is "adjacent" to more than two $T_{\sigma'}$'s, and is separated from them by a collection of disks $\{D_\tau : \tau$ is a vertex of $\sigma\}$, where there is one D_τ for each $T_{\sigma'}$.

The rest of the proof is the same as the proof for Theorem 1 with the appropriate change in notation.

3. In this section we take the neighbourhood N of Theorem 1 and modify it so that the simple closed curve J (respectively, graph G) is homotopic in N to a homeomorphic copy of itself in ∂N. As a consequence we can rename subsets of the new neighbourhood so that it satisfies the conclusion of Theorem 1 (respectively, Theorem 1′) except that possibly (5) holds in a slightly weaker form, but in which J (respectively, G) is homotopic in N to a spine of T. There is a corresponding version of each theorem for non-orientable 3-manifolds, which holds with little or no change in proof.

THEOREM 2. *Let J be a simple closed curve in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, J has a cube-with-handles neighbourhood*

$N = T \cup H_1 \cup H_2 \cup \ldots \cup H_n$ *as given in Theorem* 1 *with the additional property that* J *is* ϵ*-homotopic in* N *to a simple closed curve* L *on* ∂N *which crosses each* D_i *precisely once and has no other points of intersection with* $\bigcup D_i$.

Proof. Let N be a cube-with-handles neighbourhood of J as given by Theorem 1 for $\epsilon/3$. Denote $T_i \cup H_i$ by N_i. Then $J_i \subset T_{i-1} \cup N_i \cup T_{i+1}$, $\operatorname{diam}(T_{i-1} \cup N_i \cup T_{i+1}) < \epsilon/3$, $p_{i-1} \in \operatorname{Int} D_{i-1}$, $p_i \in \operatorname{Int} D_i$, where p_{i-1}, p_i are the endpoints of J_i, and D_i is the disk $N_i \cap N_{i+1} = T_i \cap T_{i+1}$.

Let us consider $N_i = T_i \cup H_i$. Let $E_1, E_2, \ldots, E_k$ be a collection of handles for H_i. That is, $E_1, E_2, \ldots, E_k$ is a mutually exclusive collection of disks properly embedded in H_i such that the closure of H_i minus a sufficiently close regular neighbourhood of $\bigcup E_j$ is a 3-cell. By choosing the E_j to miss the disk $F_i = T_i \cap H_i$, we ensure that $\bigcup E_j$ is a collection of handles for the cubes-with-handles $T_{i-1} \cup N_i \cup T_{i+1}$ and also for the cube-with-handles N_i.

Let $\delta > 0$ be less than half the distance between any two of the disks $E_1, E_2, \ldots, E_k$. Let J_i'' be any polygonal arc which is δ-homotopic to J_i in $T_{i-1} \cup N_i \cup T_{i+1}$ by a homotopy keeping p_{i-1} and p_i fixed. Suppose that J_i and J_i'' are oriented from p_{i-1} to p_i and that each E_j is oriented. Suppose further that J_i'' pierces each disk E_j at each point of intersection.

We assign a letter to each crossing of J_i'' with one of the disks E_j as follows:

e_j is a positive crossing through the disk E_j,
e_j^{-1} is a negative crossing through the disk E_j.

Using this convention we can write out a word in the letters $e_1, e_2, \ldots, e_k$ representing J_i''. For example, if J_i'' were represented by the word $e_1e_3^{-1}e_2$ it would mean that proceeding along J_i'' from p_{i-1} to p_i, J_i'' crosses E_1 in the positive direction, then E_3 in the negative direction, then E_2 in the positive direction and that there are no other intersections of J_i'' with any E_j.

If J_i'' is represented by the word

$$x_1^{\epsilon_1}x_2^{\epsilon_2}\ldots x_l^{\epsilon_l}$$

where $\epsilon_j = \pm 1$ and $x_j = e_r$ for some $r \in \{1, 2, \ldots, k\}$, we can obtain (uniquely) a reduced word

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2}\ldots\alpha_m^{\eta_m}$$

where $\eta_j = \pm 1$, α_j is some e_r, and no symbol $\alpha_j^{\eta_j}\alpha_{j+1}^{\eta_{j+1}}$ is of the form $\alpha^{-1}\alpha$ or $\alpha\alpha^{-1}$. We do this by successively deleting such combinations from the word $x_1^{\epsilon_1}x_2^{\epsilon_2}\ldots x_l^{\epsilon_l}$ until none occur. (If J_i'' were a closed curve at p_{i-1} this procedure would yield the reduced word of J_i'' in the fundamental group based at p_{i-1} of the cube-with-handles $T_{i-1} \cup N_i \cup T_{i+1}$ in a presentation of this group.) Note that for any two approximations to J_i such as J_i'' we obtain the same reduced word which we will refer to as the word of J_i. This follows from our choice of δ.

Now suppose that $\alpha_1^{\eta_1}\alpha_2^{\eta_2}\ldots\alpha_m^{\eta_m}$ is the (reduced) word of J_i. We emphasize that this word is unique. We would like to construct an arc on

$\partial N_i - (D_{i-1} \cup D_i)$ whose word is also $\alpha_1^{\eta_1}\alpha_2^{\eta_2} \dots \alpha_m^{\eta_m}$. In general, this is not possible. However, there is an oriented polygonal curve L' in ∂N_i running from a point p'_{i-1} in ∂D_{i-1} to a point p_i' in ∂D_i with $\operatorname{Int} L' \subset \partial N_i - (D_{i-1} \cup D_i)$ such that the word of L' is the same as that of J_i and L' has a finite number of self-intersections, all crossing points. We may assume that each such crossing is a double point and that none of the double points lies in the boundary of one of the disks E_j, $j = 1, 2, \dots, k$. We will drill out holes in N_i to make a new cube-with-handles in N_i for which it is possible to find an L' with no self-intersections.

Let q denote one of the double points of L' and let A', A'' be two subarcs of L' which cross at q and contain no other singularities of L. Let A' and A'' have the orientations inherited from L'. If E is a small disk in N_i with $E \cap \partial N_i = \partial E \cap \partial N_i = A'$ and $E \cap (\bigcup E_j) = \emptyset$, then we could drill out a tube in N_i along the arc $A = \partial E - \operatorname{Int} A'$ and obtain a new cube-with-handles $N_i' \subset N_i$ in place of N_i. By replacing the subarc A' of L' with the arc A on the tube we can reduce the number of singularities of L' by one. However, the new curve L'' so obtained would have a different word in N_i' because of the crossing of A'' through the disk E, which must now be taken as a handle of $T_{i-1} \cup N_i' \cup T_i$ along with $E_1, E_2, \dots, E_k$.

To compensate for this we choose E so that J_i also has this letter in its (reduced) word. Suppose that the new letter e corresponding to passage of L'' through E is between $\alpha_s^{\eta_s}$ and $\alpha_{s+1}^{\eta_{s+1}}$ in the word of L'. The word of L'', then, is the word of L' with e inserted between $\alpha_s^{\eta_s}$ and $\alpha_{s+1}^{\eta_{s+1}}$:

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2} \dots \alpha_s^{\eta_s} e \alpha_{s+1}^{\eta_{s+1}} \dots \alpha_m^{\eta_m}.$$

J_i can be divided into 3 subarcs B_1, B_2, B_3 such that the word of B_1 is $\alpha_1^{\eta_1}\alpha_2^{\eta_2} \dots \alpha_s^{\eta_s}$, the word of B_2 is the identity (i.e., B_2 does not intersect any of the disks $E_1, E_2, \dots, E_k$), and the word of B_3 is $\alpha_{s+1}^{\eta_{s+1}} \dots \alpha_m^{\eta_m}$.

To obtain the arcs B_1, B_2 and B_3 we first take a regular neighbourhood U of $\bigcup E_j$ in N_i missing the endpoints of J_i. There is a finite collection of mutually disjoint closed subintervals of J_i, the union of the interiors of which cover $J_i \cap (\bigcup E_j)$, such that each interval lies in U. We can read a word for J_i from these intervals as follows: If the endpoints of an interval are separated in U by some E_k, then that interval represents a (net) crossing of J_i through E_k in either a positive direction or a negative direction. In the first case we associate the letter e_k with the interval; in the second case, the letter e_k^{-1}. If the endpoints of the interval are not separated in U by any E_k, then that interval represents a (net) crossing of J_i through $\bigcup E_k$ of zero. If the interval intersects some E_k (it can intersect at most one), associate the letter t with that interval; otherwise, just eliminate it from the collection. The word for J_i is the word obtained by traversing J_i from p_{i-1} to p_i writing down the letter associated with each interval as we come to it. It follows that, if we treat t as equal to the trivial word, then this word reduces to the (reduced) word of J_i obtained before. Furthermore, reduction can be accomplished geometrically by replacing the two intervals corresponding to an $e_k e_k^{-1}$ (or an $e_k^{-1} e_k$) by a longer interval equal to the union of these two intervals with the subinterval of J_t lying between them, associating the letter t with it (and ignoring it, henceforth, in the reduction as it represents the trivial word). It follows that the (reduced) word of J_t is represented by some subcollection of the original collection of intervals. We now have a collection $\mathscr{A}$ of closed intervals of J_t remaining (including the intervals associated with letter t). Now eliminate from $\mathscr{A}$ any interval which is contained in some other interval in $\mathscr{A}$. Note that each of these eliminated intervals has the letter t associated with it. Then $\mathscr{A}$ is a collection of mutually disjoint closed intervals and the union of the interiors of these intervals covers $J_t \cap (\bigcup E_j)$. Furthermore, the (reduced) word of J_t can be read directly from the intervals of $\mathscr{A}$ if letter t is ignored in each occurrence. Therefore there is a point x of J_t such that the

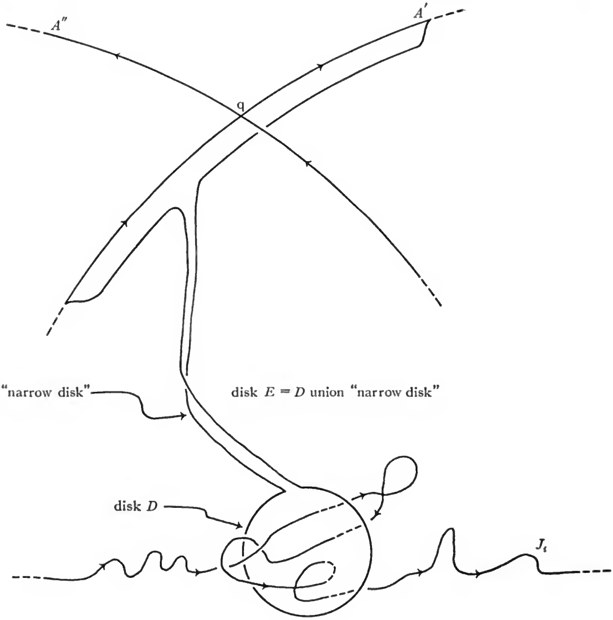

FIGURE 1

word of J_i from p_{i-1} up to x is the word $\alpha_1^{\eta_1} \dots \alpha_s^{\eta_s}$ and x does not lie in any of the intervals of $\mathscr{A}$. Let B_2 be a subarc of J_i containing x which is disjoint from each interval of $\mathscr{A}$. Let B_1 and B_3 be the closures of the appropriate components of $J_i - B_2$.

Let D be a small disk in $\operatorname{Int} N_i$ missing $\bigcup E_j \cup (J - \operatorname{Int} B_2)$ such that any sufficiently close polygonal approximation to J intersects D algebraically once. Let β be a polygonal arc joining ∂D to $q(=A' \cap A'')$ with $\operatorname{Int} \beta \subset \operatorname{Int} N_i - (J \cup (\bigcup E_j))$. That β can be chosen to miss $\bigcup E_j$ follows because $N_i - \bigcup E_j$ is homeomorphic to a 3-cell less a finite disjoint collection of disks in its boundary. Replace β with a narrow disk (see Figure 1) intersecting L' in the subarc A' and D in a subarc of ∂D. Let the disk E be the union of D and the narrow disk. Drill a small tubular hole out of N_i along the arc $\partial E - A'$ to obtain the cube-with-handles N_i'. A set of handles for N_i' is $E, E_1, \dots, E_k$. Let L'' be the arc obtained from L' by replacing the subarc A' with an arc running along the tube and not intersecting E. If the narrow disk is given the appropriate "twist" before attaching to D to form E, then J_i has the same word

$$\alpha_1^{\eta_1}\alpha_2^{\eta_2} \dots \alpha_s^{\eta_s} e \alpha_{s+1}^{\eta_{s+1}} \dots \alpha_m^{\eta_m}$$

in N_i' as does L''.

It is clear that we can apply the above technique at each point of singularity of L' and obtain a cube-with-handles $N_i' \subset N_i$ by drilling out small tubular holes in N_i. We can replace the singular arc L' by a non-singular arc L_i lying in $\partial N_i' - (D_{i-1} \cup D_i)$ except for its endpoints $p'_{i-1} \in \partial D_{i-1}$ and $p_i' \in \partial D_i$. Furthermore, the construction gives a set of handles $E_1^i, E_2^i, \dots, E_{k_i}^i$ for N_i' consisting of the handles for N_i together with the handles such as E introduced at the points of singularity of L'. The word of L_i in N_i' is the same as the word of J_i. Notice also that $J - \operatorname{Int} J_i$ is disjoint from each handle E_j^i.

Let $L = \bigcup \{L_i : i = 1, 2, \dots, n\}$. Then L is a simple closed curve on $\partial N' = \partial(\bigcup N_i')$. All that remains is to show that L and J are ϵ-homotopic in $N' = \bigcup N_i'$.

Join p_i to p_i' by an arc l_i in D_i and orient l_i from p_i to p_i'. Then $J_i l_i L_i^{-1} l_{i-1}^{-1}$ is an oriented loop in $N'_{i-1} \cup N_i' \cup N'_{i+1}$ based at p_i. Because this loop does not intersect the handles of N'_{i-1} or of N'_{i+1} and its word in N_i' is zero, this loop represents the trivial word in $N'_{i-1} \cup N_i' \cup N'_{i+1}$ and thus bounds a singular disk in this cube-with-handles. By piecing together these singular disks, we obtain an ϵ-homotopy in N' from J to L.

Dropping the primes, we have the structure required in Theorem 2.

COROLLARY 1. *Let J be a simple closed curve in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, J has a cube-with-handles neighbourhood $N = T \cup H_1 \cup H_2 \cup \dots \cup H_n$ as given in Theorem* 1, *except that conclusion* (5) *becomes:*

(5) *there are 3-cells $C_i \subset N_i$ with $C_i \cap \partial N_i \supset D_{i-1} \cup D_i$ and $J_i \subset C_{i-1} \cup N_i \cup C_{i+1}$ (where $N_i = T_i \cup H_i$),*

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and with the additional property that J is ϵ-homotopic in N to a geometric centre-line of T.

Proof. Let $N = N_1 \cup N_2 \cup \ldots \cup N_n$ be the neighbourhood constructed in Theorem 2 for J. Let T_i be a regular neighbourhood in N_i of $L_i \cup D_{i-1} \cup D_i$ and let H_i be the closure of $N_i - T_i$. Let $T = \bigcup T_i$.

Let the 3-cells C_i be N_i minus a regular neighbourhood of $\bigcup E_j^i$ which is so close to $\bigcup E_j^i$ that it is disjoint from $J - \operatorname{Int} J_i$.

THEOREM 2′. *Let G be a finite graph topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood $N = T \cup (\bigcup H_\sigma)$ as given by Theorem 1′ with the additional property that J is ϵ-homotopic in N to a polygonal finite graph L in ∂N which is homeomorphic to G.*

Proof. The proof is essentially the same as that of Theorem 2. If P is the special decomposition of G used in constructing a neighbourhood N as in Theorem 1′, each 1-element σ of P has an associated cube-with-handles $N_\sigma = T_\sigma \cup H_\sigma$. We drill tubes out of each N_σ to form an N_σ' which is a cube-with-handles and construct a homotopy of σ in $\bigcup \{N_{\sigma'} : \sigma'$ is a 1-element of P and $\sigma' \cap \sigma \neq \emptyset\}$ onto a copy of σ in $\partial N_\sigma'$ as in the proof of Theorem 2. The main difference occurs when σ is an n-frame.

Suppose that σ is an n-frame of P with $n > 2$. That is, σ is a 1-element of P containing a point v of G of order greater than 2. Let $\tau_1, \tau_2, \ldots, \tau_m$ be the vertices of σ. Let $E_1, E_2, \ldots, E_k$ be a set of handles for N_σ. Choose an arc γ in $N_\sigma - (G - \{v\}) - \bigcup E_j$ joining v to a point v' of

$$\partial N - \bigcup \{D_{\tau_i} : i = 1, 2, \ldots, m\}.$$

The n-frame σ is the union of 1-simplexes σ_i, $i = 1, 2, \ldots, m$, such that one vertex of σ_i is v and the other is τ_i. Construct, as in Theorem 2, singular arcs L'_{σ_i} on $\partial N_\sigma - \bigcup \operatorname{Int} D_{\tau_i}$ joining v' to a point $\tau_i' \in \partial D_{\tau_i}$, such that the word of L'_{σ_i} is the (reduced) word of σ_i in N_σ. Each L'_{σ_i} may cross itself and other L'_{σ_j}'s as well. We can drill tubes in N_σ as before to obtain a new N_σ' on which each L'_{σ_i} can be replaced by a non-singular L_{σ_i} such that $L_{\sigma_i} \cap L_{\sigma_j}$ is the point v' and the word of L_{σ_i} in N_σ' is the same as the word of σ_i in N_σ'. To do this involves only a simple generalization of the technique of Theorem 2 to the case of a finite collection of arcs.

If l_i is an arc in D_{τ_i} from τ_i to τ_i' which misses $G - \{\tau_i\}$, then $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$ is a simple loop which lies in N_σ' except for part of σ_i. The part of σ_i outside of N_σ' does not intersect any handles of any $N'_{\sigma'}$, $\sigma' \neq \sigma$, so the word of $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$ in

$$\bigcup \{N_{\sigma'}' : \sigma' \cap \sigma \neq \emptyset\}$$

is the same as its word in N_σ', namely zero. Thus it bounds a singular disk in $\bigcup \{N'_{\sigma'} : \sigma' \cap \sigma \neq \emptyset\}$. Piecing together along γ all the singular disks obtained in this way (one for each $\sigma_i l_i L_{\sigma_i}^{-1}\gamma^{-1}$), we obtain an ϵ-homotopy in N_σ' of $\sigma = \bigcup \sigma_i$ onto $\bigcup L_{\sigma_i}$ in $\partial N_\sigma'$. Let $L_\sigma = \bigcup L_{\sigma_i}$.

By piecing together all the L_σ's, we obtain a homeomorphic copy L of G on the boundary of $N' = \bigcup \{N_\sigma' : \sigma \text{ is a 1-element of } P\}$. By piecing together the homotopies, we obtain an ϵ-homotopy of G onto L in N'.

COROLLARY 2. *Let G be a finite graph topologically embedded in the interior of an orientable 3-manifold M^3. For any $\epsilon > 0$, G has a cube-with-handles neighbourhood $N = T \cup (\bigcup H_\sigma)$ as given in Theorem 1′ except that conclusion* (5) *becomes:*

(5) *there exist 3-cells $C_\sigma \subset N_\sigma$ with $C_\sigma \cap \partial N_\sigma \supset \bigcup \{D_\tau : \tau \text{ is a vertex of } \sigma\}$ and $\sigma \subset N_\sigma \cup (\bigcup \{C_{\sigma'} : \sigma' \text{ is a 1-element of } P \text{ and } \sigma' \cap \sigma = \emptyset\})$ (where $N_\sigma = T_\sigma \cup H_\sigma$),*

and with the additional property that G is ϵ-homotopic in N to a 1-spine of T.

Proof. Let $N = \bigcup N_\sigma$ be the neighbourhood constructed in Theorem 2′ for G. Let T_σ be a regular neighbourhood of $L_\sigma \cup (\bigcup \{D_\tau : \tau \text{ is a vertex of } \sigma\})$ in N_i and let H_σ be the closure of $N_\sigma - T_\sigma$. Let $T = \bigcup \{T_\sigma : \sigma \text{ is a 1-element of } P\}$. Then the T_σ and H_σ give N the required structure.

The 3-cell C_σ is N_σ minus a regular neighbourhood of the handles of N_σ which is sufficiently close to these handles to not intersect $G - \operatorname{Int} \sigma$.

4. A second smaller neighbourhood. Let M^3 be an orientable 3-manifold, and let J be a simple closed curve in Int M^3 which is homologous to zero in M^3. In this section, we will take the neighbourhood N of J given in Theorem 1, and construct a smaller neighbourhood N^1 which lies "nicely" in N. We will also construct a spanning surface S in $N - \operatorname{Int} N^1$. This will be the inductive step in constructing an infinite sequence of neighbourhoods in the next section.

Let S be a polyhedral surface in a 3-manifold, and let δ be a polydehral arc which intersects S only in its endpoints. There is a 3-cell B such that $B \cap S$ consists of two disks D_1 and D_2 on ∂B, $\operatorname{Int} \delta \subset \operatorname{Int} B$, and the endpoints of δ are in $\operatorname{Int} D_1$ and $\operatorname{Int} D_2$, respectively. We can now add a handle to S by replacing $(\operatorname{Int} D_1) \cup (\operatorname{Int} D_2)$ with $\partial B - (D_1 \cup D_2)$. We call this operation *adding a handle to S along δ*. Note that if S is orientable and two-sided, and if δ approaches S on the same side at both endpoints of δ, then the handle added to S is orientable.

Step 1. *Let N be a cube-with-handles neighbourhood of J as given in Theorem* 1. *Then there is an orientable surface $S^0 \subset M^3 - \operatorname{Int} N$ where $S^0 \cap N = \partial S^0 = L$ is a simple closed curve which is homologous to J in N. Furthermore, S^0 can be chosen so that $\partial S^0 = L$ intersects every disk D_i exactly once.*

Remark. If J is not homologous to zero in M^3, there is still a simple closed curve L on ∂N so that J is homologous to L in N and such that L intersects each D_i exactly once.

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Proof. As in the proof of Theorem 1, we can choose a polygonal approximation J' to J which is homotopic to J in Int N. Furthermore, we can assume that the points $p_1, p_2, \ldots, p_n$ are on J' and that they divide J' into subarcs $J_1', J_2', \ldots, J_n'$ where $J_i' \subset T_{i-1} \cup T_i \cup H_i \cup T_{i+1}$ (mod n), and where J' pierces the disk D_i at the point p_i. It is now easy to see that J' intersects each D_i algebraically once.

Since J' is homologous to zero in M^3, J' bounds an orientable (and hence two-sided) surface S' in Int M^3. Suppose that S' is in general position with respect to ∂N and each ∂D_i. Then $S' \cap \partial N$ is a 1-cycle in ∂N which intersects each ∂D_i algebraically once on ∂N. If $S' \cap \partial D_i$ contains more than one point, there is a subarc δ_i of ∂D_i which intersects S' only in its endpoints. Furthermore, δ_i can be chosen so that it approaches S' on the same side at both endpoints. Thus, we can add an orientable handle to S' along δ_i. By adding handles of this type, we can insure that $S' \cap \partial N$ intersects each ∂D_i exactly once.

Let $N_i = T_i \cup H_i$. Then each N_i is a cube-with-handles, $N = \bigcup N_i$, and $N_i \cap N_{i+1} = D_i$. Thus $S' \cap \partial N_i \cap \partial N$ is now an arc ξ_i from ∂D_{i-1} to ∂D_i, plus a finite collection of simple closed curves missing D_{i-1} and D_i. If there are any such simple curves in $S' \cap \partial N_i \cap \partial N$, there is an arc δ_i' from one of them to ξ_i on $\partial N_i \cap \partial N$. The arc δ_i' can be chosen so that it approaches S' on the same side at both endpoints. Then we can add an orientable handle to S' along δ_i', and this will reduce the number of simple closed curves of $S' \cap \partial N_i \cap \partial N$ by one. In this way, we can insure $S' \cap \partial N$ is one simple closed curve which intersects each D_i exactly once. Let

$$S^0 = S' \cap (M^3 - \text{Int } N).$$

Step 2. *Let N be a neighbourhood of J as given in Theorem* 1 *and let* $\epsilon' > 0$. *Then there is a neighbourhood* N^1 *of J in* Int N *with*

$$N^1 = (\bigcup T_j^1) \cup (\bigcup H_j^1)$$

where T_j^1, H_j^1, J_j^1, p_j^1, *and* D_j^1 *are as described in Theorem* 1. *Furthermore, if* $p_i = J_i \cap J_{i+1}$ (mod n), *then for some* $j = 1, 2, \ldots, n_1$, $p_i = p_j^1 = J_j^1 \cap J^1{}_{j+1}$ (mod n_1). *Also, each* D_i *can be adjusted in a neighbourhood of* ∂T^1 *so that*

$$p_i = p_j^1 \subset \text{Int } D_j^1 \subset D_j^1 \subset \text{Int } D_i.$$

Proof. We repeat the construction of Theorem 1 to construct N^1. The points $p_1^1, \ldots, p_{n_1}{}^1$ can be chosen so that each p_i is a p_j^1. Thus, if $p_j^1 = p_i$, the disk D_j^1 can be chosen initially so that it is a subdisk of D_i. For each adjustment of D_j^1 near ∂T^1 in the construction of Theorem 1, D_i can also be adjusted in the same way near ∂T^1 so that D_j^1 remains a subdisk of D_i.

Step 3. *Given neighbourhoods N and* N^1 *as in Step* 2, *there is a disjoint collection of orientable surfaces* $E_1, \ldots, E_n$ *such that* $E_i \cap \partial N = \partial D_i$, *and* $E_i \cap N^1 = \partial D_j^1$ (*where* D_j^1 *is the special subdisk of* D_i *defined in Step* 2). *Each* E_i *can be obtained by adding handles to the annulus* $D_i -$ Int D_j^1.

Proof. Let J' be a polyhedral centreline for $T^1 \subset N^1$ so that each $p_i \in J'$ and is in general position with respect to each D_i. As in § 2, we can associate a word with the intersections of J' and $D_1, D_2, \ldots, D_n$. Thus for each disk D_i we have a letter e_i. Each time J' crosses D_i in a positive direction, the letter e_i appears in the words, and for each negative crossing of D_i, the letter e_i^{-1} appears. We consider this word a cyclic word; in other words, it is equivalent to any of its cyclic permutations. Since J' is homotopic in N to a simple closed curve which pierces each D_i exactly once, this word freely reduces to the word $e_1e_2 \ldots e_n$. Corresponding to each free reduction $e_ie_i^{-1}$ (or $e_i^{-1}e_i$) we can add an orientable handle to D_i. In this way, we obtain new surfaces, also called $D_1, D_2, \ldots, D_n$ so that $J' \cap D_i = p_i$. Since J' is a spine for T^1, there is an isotopy of N onto itself, fixed on ∂N, which pushes each D_i off T^1, except for the disks $D_j^1 \subset D_i$ (where D_j^1 is the meridional disk of T^1 containing $p_j^1 = p_i$).

For each $j = 1, 2, \ldots, n$ there is a wedge of simple closed curves in H_j^1 so that this wedge is a spine of H_j^1. We can assume that the wedge lies in the interior of H_j^1, except for the wedge point which lies in the interior of the disk $F_j^1 = H_j^1 \cap T^1$. Again, we can add orientable handles to the D_i's so that they do not intersect the wedge. Then there is an isotopy of N onto itself which pushes the D_i's off H_j^1. Therefore, we can assume that $D_i \cap N^1 = D_j^1$. Let

$$E_i = D_i - \operatorname{Int} D_j^1.$$

If N^1 is chosen sufficiently close to J, we can insure that each annulus with handles E_i constructed in this step lies in the union of the sections N_{i-1}, N_i, N_{i+1}, and N_{i+2} of the original neighbourhood N.

Step. 4. *Let N and N^1 be neighbourhoods of J as in Steps* 2 *and* 3. *Let L be a simple closed curve in ∂N which is homologous to J in N. Then there is an orientable surface $S \subset N - \operatorname{Int} N^1$ such that $S \cap \partial N = L$, $S \cap \partial N^1$ is a simple closed curve L^1 which is homologous to J in N^1, and $\partial S = L \cup L^1$. Furthermore, S can be chosen so that $S \cap E_i$ is an arc joining L to L^1.*

Proof. Let J'' be a polyhedral simple closed curve in N^1 which is homologous to J in N^1. Then L is homologous to J'' in N, so there is a surface S' such that $\partial S' = L \cup J''$. By the proof of Step 1 we can assume that $S' \cap \partial N^1$ is a simple closed curve L^1 which intersects each D_j^1 exactly once. Let $S = S' \cap (N - \operatorname{Int} N^1)$.

For each i, $S \cap E_i$ is an arc ξ_i joining the two boundary components of E_i, plus a finite number of simple closed curves. If this number of simple closed curves in $S \cap E_i$ is non-zero, there is an arc δ_i joining one of them to the arc ξ_i. The arc δ_i can be chosen so that it approaches S on the same side at both endpoints. We can add an orientable handle to S along δ_i, and this will reduce the number of simple closed curves in $S \cap E_i$ by one. Thus, we can assume that for each i, $S \cap E_i$ is an arc joining L to L^1.

Step 5. *Let K_i be the closure of the component of $N - (N^1 \cup (\bigcup_{i=1}^n E_i))$ such that $E_{i-1} \cup E_i \subset \operatorname{Cl}(K_i)$. Then K_i is a* 3*-manifold with connected boundary,* *and* $S_i = S \cap K_i$ *is an orientable surface with connected boundary which is properly embedded in* K_i, *and* ∂S_i *does not separate* ∂K_i. *Furthermore,* diam $K_i < \epsilon$.

Proof. This step just restates the results of the previous steps.

5. An infinite sequence of neighbourhoods.

In Theorem 3 we construct an infinite sequence of cubes-with-handles neighbourhoods of the simple closed curve J, and an open surface S whose closure is $S \cup J$. In Theorem 3′, we construct a similar sequence of neighbourhoods for a finite graph.

The proof of Theorem 3 is contained in Steps 1–5 of the previous section.

THEOREM 3. *Let* M^3 *be an orientable 3-manifold, and let* J *be a simple closed curve in* Int M^3 *which is homologous to zero in* M^3. *Then there exist cubes-with-handles* $N^1, N^2, N^3, \dots$ *and an open surface* S *such that:*

(1) $\text{Int } M^3 \supset N^1 \supset \text{Int } N^1 \supset N^2 \supset \text{Int } N^2 \supset \dots \supset J$ *and* $J = \bigcap_{k=1}^{\infty} N^k$.

(2) $N^k - \text{Int } N^{k+1} = K_1^k \cup \dots \cup K^k_{n_k}$ *where each* K_i^k *is a cube-with-holes.*

(3) $K^k_{i+1} \cap K_i^k = E_i^k$ *where* E_i^k *is an annulus with orientable handles with one boundary component contained in* ∂N^k *and the other boundary component contained in* ∂N^{k+1}.

(4) $K_i^k \cap \partial N^k = \alpha_i^k$ *where* α_i^k *is an annulus with orientable handles.*

(5) $K_i^k \cap \partial N^{k+1} = \beta_i^k$ *where* β_i^k *is an annulus with orientable handles.*

(6) $\partial K_i^k = E^k_{i-1} \cup E_i^k \cup \alpha_i^k \cup \beta_i^k$.

(7) $S = S^0 \cup S^1 \cup S^2 \cup S^3 \cup \dots$, *where, for each* $k \neq 0$, $S^k \subset N^k - \text{Int } N^{k+1}$ *is an annulus with orientable handles. One boundary component of* S^k *is contained in* ∂N^k *and one boundary component of* S^k *is contained in* ∂N^{k+1}. *The surface* $S^0 \subset M^3 - \text{Int } N^1$ *is a disk with handles, and* $\partial S^0 \subset \partial N^1$.

(8) $S^k \cap K_i^k = S_i^k$ *is a disk with orientable handles properly embedded in* K_i^k. *Furthermore,* ∂S_i^k *is made up of a spanning arc of* E^k_{i-1}, *a spanning arc of* α_i^k, *a spanning arc of* E_i^k, *and a spanning arc of* β_i^k. *(Thus,* ∂S_i^k *does not separate* ∂K_i^k.*)*

(9) *There exist points* $p_i^k, \dots, p^k_{nk}$ *on* J *dividing* J *into segments* $J_1^k, \dots, J^k_{nk}$ *with* $p_i^k = J_i^k \cap J^k_{i+1}$ (mod n_k).

(10) *Each* J_j^{k+1} *is contained in some* J_i^k, *and each* α_j^{k+1} *is contained in some* β_i^k.

(11) *If* $p_i^k = p_j^{k+1}$, *then* $E_i^k \cap E_j^{k+1}$ *is a simple closed curve in* ∂N^{k+1}.

(12) $\text{diam}(K_i^k \cup J_i^k) < 1/k$.

Definition. Let $J_1, J_2, \dots, J_n$ be a collection of mutually exclusive simple closed curves in a space X. Let S be an open orientable surface in X with $S \cap (\bigcup J_j) = \emptyset$ and $\text{Cl}\, S = S \cup (\bigcup J_i)$. We say that $\bigcup J_j$ *bounds the open surface* S if there is a sequence $h_1, h_2, \dots$ of disjoint disks with handles in S with the following properties:

(1) diam $h_i \to 0$ as $i \to \infty$

(2) $S - \bigcup h_i$ contains no non-separating simple closed curves.

Note that if $h_1, h_2, \ldots$ is a finite sequence, then $\mathrm{Cl}(S)$ is a surface whose boundary is $\bigcup J_j$.

COROLLARY 3. *Let J be a simple closed curve topologically embedded in the interior of a 3-manifold M^3, and suppose that J is homologous to zero in M^3. Then J bounds an open surface S in M^3.*

Proof. If M^3 is orientable this is part of Theorem 3. The required open surface is $S = S^0 \cup S^1 \cup S^2 \cup \ldots$ and the null sequence of disks with handles are the S_i^k with a small annulus about ∂S_i^k removed to make them disjoint. If M^3 is non-orientable, Theorem 1 still is valid if T is allowed to be a solid Klein bottle. The construction of the sequence of neighbourhoods and the surface proceeds analogously as in Steps 1–5 of § 3 and Theorem 3.

Question. What are necessary and sufficient conditions for a simple closed curve to be the boundary of a compact surface?

COROLLARY 4. *Let J_1 and J_2 be disjoint simple closed curves topologically embedded in the interior of a 3-manifold M^3 with J_1 homologous to J_2 in M^3. Then $J_1 \cup J_2$ bounds an open surface S in M^3.*

Remark. By virtue of Conclusion (8) of Theorem 1, S may be chosen so that if $p \in J$ (respectively, $p \in J_1$ or $p \in J_2$) is a point at which the simple closed curve is locally tame, then the null sequence $h_1, h_2, \ldots$ of disks with handles of S does not cluster at p. In fact, $\lim_{i\to\infty} h_i$ lies in the set of wild points of J (respectively, $J_1 \cup J_2$). Thus if J (respectively, $J_1 \cup J_2$) is tame, then $h_1, h_2, \ldots$ is a finite sequence and $\mathrm{Cl}\, S$ is a surface whose boundary is J (respectively, $J_1 \cup J_2$).

COROLLARY 5. *Let J be a simple closed curve in the interior of a 3-manifold M^3 and let $p \in J$. Then there is a connected non-compact surface E with one simple closed curve boundary component such that* $\mathrm{Cl}(E) = E \cup p$, $E \cap J = \emptyset$, *and* $\mathrm{Cl}(E)$ *locally separates J at p.*

Proof. In the construction of the neighbourhood sequence in § 3, choose p to be a p_i^k. By Conclusion 11 of Theorem 3, $E = \bigcup \{E_j^l : p_j^l = p_i^k, l \geqq k\}$ is the required non-compact surface.

THEOREM 3′. *Let G be a finite graph topologically embedded in an orientable 3-manifold M^3. Then there exist cubes-with-handles $N^1, N^2, N^3, \ldots$ such that:*

(1) $\mathrm{Int}\, M^3 \supset N^1 \supset \mathrm{Int}\, N^1 \supset N^2 \supset \mathrm{Int}\, N^2 \supset \ldots \supset G$ *and* $G = \bigcap_{k=1}^{\infty} N^k$,

(2) *There is a sequence $P^1, P^2, P^3, \ldots$ of special decompositions of G so that each P^k is a subdivision of P^{k-1}.*

(3) *For each 1-element σ of P^k, there is an associated cube-with-holes K_σ^k.*

(4) $N^k - \mathrm{Int}\, N^{k+1} = \bigcup \{K_\sigma^k :$ *σ is a 1-element of P^k*$\}$.

(5) *If σ and σ' are two one elements of P^k which intersect in a vertex τ, there is an annulus with orientable handles E_τ^k so that $K_\sigma^k \cap K_{\sigma'}^k = E_\tau^k$. If $\sigma \cap \sigma' = \emptyset$, then $K_\sigma^k \cap K_{\sigma'}^k = \emptyset$.*

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(6) *E_τ^k is properly embedded in $N^k - \operatorname{Int} N^{k+1}$. One component of ∂E_τ^k is contained in ∂N^k, and one component is contained in ∂N^{k+1}.*
(7) *If τ is a vertex of both P^k and P^{k+1}, then $E_\tau^k \cap E_\tau^{k+1} \subset \partial N^{k+1}$ is a simple closed curve.*
(8) *If σ is a 1-element of P^k, then $\operatorname{diam}(K_\sigma^k \cup \sigma) < 1/k$.*
(9) *If τ is a vertex of P^k, $\bigcup_{i=k}^{\infty} E_\tau^i$ is a noncompact surface E_τ with one boundary component in ∂N^k, and whose closure is $E_\tau \cup \tau$.*

Proof. The proof of Theorem 3′ is analogous to the proof of Theorem 3.

6. Constructing the monotone map which tames J.

LEMMA 1. *Let K be an orientable compact 3-manifold with connected boundary, and let S be a disk with orientable handles properly embedded in K so that ∂S does not separate ∂K. Let H be a solid torus, and let F be a handle for H (i.e., F is a non-separating properly embedded disk in H). Let f_0 be a monotone map of ∂K onto ∂H and f_1 be a monotone map of S onto F where each of the finite number of nondegenerate point inverses of f_0 and f_1 is a finite 1-complex missing ∂S, and where $f_0|\partial S = f_1|\partial S$. Then f_0 and f_1 can be extended to a monotone map f from K onto H such that $f(\operatorname{Int} K) = \operatorname{Int} H$. Furthermore, suppose X is a compact set in $\operatorname{Int} K - S$ with the following property: For each open set $U \subset \operatorname{Int} K$, either $U - (U \cap X)$ is connected or $(\operatorname{Bd} U) \cap X \neq \emptyset$. Then f can be constructed so that each component of X is a point inverse.*

Remark. A similar result could be proved for any cube-with-handles H. This lemma will be used to construct a monotone mapping from each K_i^k constructed in Theorem 3 onto a solid torus.

Proof. Let $R(S)$ be an embedding of $S \times [-1, 1]$ in K with $S \times 0$ identified with S and lying so close to S that it is disjoint from the non-degenerate point inverses of f_0. Let $R(F)$ be an embedding of $F \times [-1, 1]$ in H such that $f_0(\partial K \cap R(S)) = \partial H \cap R(F)$. By using the product structures of $R(S)$ and $R(F)$, we extend f_0 and f_1 to a "level preserving" monotone map

$$f : \partial K \cup R(S) \to \partial H \cup R(F).$$

Let $K_1 = \operatorname{Cl}(K - R(S))$ and $H_1 = \operatorname{Cl}(H - R(F))$. Then $f|\partial K_1$ is a monotone map onto ∂H_1.

Finitely many point inverses of f lie in ∂K_1 and each is a finite 1-complex. Using [**4**, Lemma 4], we can extend f to take a collar (missing X) of ∂K_1 in K_1 onto a collar of ∂H_1 in H_1 so that f has precisely one point inverse on the inside of this collar in K_1 and each point inverse of f is a connected finite 1-complex. As in the proofs of [**2**, Theorems 6.2 and 7.6], f can be extended to carry K_1 minus this collar onto the 3-cell H_1 minus the collar of ∂H_1 so that f has each component of X as a point inverse. Thus f is the required monotone map of K onto H extending f_0 and f_1.

THEOREM 4. *Let M^3 be a closed orientable 3-manifold, and let J be a simple closed curve topologically embedded in M^3. If J is homologous to zero in M^3, then there is a monotone map f of M^3 onto S^3 such that:*

(1) *$f(J)$ is a tame unknotted simple closed curve in S^3.*

(2) *$f|J$ is a homeomorphism.*

(3) *$f(M^3 - J) = S^3 - f(J)$.*

Furthermore, suppose X is a compact set in $M^3 - J$ so that if U is any connected open set in M^3, either $(\text{Bd } U) \cap X = \emptyset$ or $U - (X \cap U)$ is connected. If J is homologous to zero in $M^3 - X$, then the map f can be chosen so that each component of X is a point inverse.

Remark. The point inverse of f form an upper semi-continuous decomposition of M^3 whose decomposition space is S^3 and whose natural quotient map is f.

Proof. Regard S^3 as E^3 union a point at infinity. Let $f|J$ be a homeomorphism of J onto the unit simple closed curve $\{(x, y, z) \in E^3 : z = 0$ and $x^2 + y^2 = 1\}$ in the xy-plane. Let $A = \{p \in E^3 : (p, f(J)) \leqq 1/2\}$ be a solid torus with centreline $f(J)$. If we write the torus ∂A as $J \times S^1$, then we can regard the solid torus A as the quotient space of $J \times S^1 \times [0, 1]$ obtained by collapsing the circles $\{p\} \times S^1 \times \{0\}$ to the points $f(p)$ of the centreline $f(J)$ of A. Then we have a quotient map $h : J \times S^1 \times [0, 1] \rightarrow A$ such that

(1) $h(J \times S^1 \times \{1\}) = \partial A$,

(2) $h|J \times S^1 \times (0, 1]$ is a homeomorphism onto $A - f(J)$,

(3) if $p \in J, h(\{p\} \times S^1 \times \{0\}) = f(p) \in f(J)$.

Furthermore, we can choose h such that, for $s_0 \in S^1, h(J \times \{s_0\} \times [0, 1])$ is an annulus lying to the inside of J in the xy-plane.

Now we suppose we have the neighbourhoods $N^1, N^2, N^3, \ldots$ of J constructed in Theorem 3, and we suppose $X \subset M^3 - (N^1 \cup S^0)$. Since each S_i^k is a disk with handles (see (7) and (8) of Theorem 3), f can be extended to a map, also called f, from $S \cup J$ onto the disk $\{(x, y, 0) : x^2 + y^2 \leqq 1\}$ so that S^0 goes to the disk $\{(x, y, 0) : x^2 + y^2 \leqq 1/2\}$ and S_i^k goes onto the disk $h(J_i^k \times \{s_0\} \times [1/k, 1/k + 1])$. Furthermore, f can be chosen so that each nondegenerate point inverse of f is a 1-complex lying either in the interior of an S_i^k or in the interior of S^0.

Since $\partial N^k = \bigcup_{i=1}^{n_k} \alpha_i^k$, and each α_i^k is an annulus with handles (see (4) of Theorem 3), f can be extended to take ∂N^k onto $h(J \times S^1 \times \{1/k\})$ such that $f(\alpha_i^k) = h(J_i^k \times S^1 \times \{1/k\})$ and such that each nondegenerate point inverse of $f|\partial N^k$ is a finite 1-complex in the interior of some α_i^k missing S_i^k.

Each E_i^k is an annulus with handles and f has been defined on ∂E_j^k and on the spanning arc $S^k \cap E_i^k$. Thus f can be extended to take E_i^k onto $h(\{p_i^k\} \times S^1 \times [1/k, 1/k + 1])$ so that $f|E_i^k$ has at most one nondegenerate point inverse, which is a 1-complex in Int E_i^k.

Since f has already been defined on the boundary of each K_i^k and on the spanning surface S_i^k, then f can be extended to take K_i^k monotonically onto the solid torus $h(J_i^k \times S^1 \times [1/k, 1/k + 1])$ as in Lemma 1. Thus we have defined f to take N^1 onto A as well as to take the spanning surface S^0 of $M^3 - \operatorname{Int} N^1$ onto the spanning disk $\{(x, y, 0) : x^2 + y^2 \leqq 1/2\}$ of the solid torus $S^3 - \operatorname{Int} A$. By Lemma 1, f can now be extended to $M^3 - \operatorname{Int} N^1$ to give the required map of M^3 onto S^3. This completes the proof of Theorem 4.

It follows from our use of Bing's results [**2**], that each nondegenerate point inverse of the map f constructed in Theorem 4 is either a component of X or is a finite 1-complex in $M^3 - J$. Using results of Armentrout [**1**] as restated in [**12**, Lemma 5], one can see that there is no such map $f : M^3 \to S^3$ which tames a wild simple closed curve J if each point inverse of f in some neighbourhood of J is cellular. If each point inverse of f in a neighbourhood of J is strongly acyclic over Z or Z_2, or has trivial Čech cohomology with coefficients Z or Z_2, it follows from [**12**, Corollaries 1 and 3] that each point inverse of f in some neighbourhood of J is cellular. Also, if the image of the nondegenerate point inverses is 0-dimensional in S^3, then it follows from [**12**, Theorem 7] that each point inverse of f in some neighbourhood of J is cellular.

COROLLARY 6. *Let J be a simple closed curve which is topologically embedded in the interior of a 3-manifold M^3. Suppose that J has a solid torus neighbourhood N in M^3 so that J is homologous to a centreline of N. Then there is a monotone map f from M^3 onto itself such that:*

(1) *$f|J$ is a homeomorphism.*

(2) *$f|M^3 - \operatorname{Int} N$ is a homeomorphism.*

(3) *$f(M^3 - J) = M^3 - f(J)$.*

(4) *$f(J)$ is tame in M^3.*

Proof. There is a neighbourhood N^1 of J in $\operatorname{Int} N$ satisfying the requirements of Theorem 1. By the techniques of Steps 2 and 3 of § 3, there is an annulus with orientable handles E properly embedded in $N - \operatorname{Int} N^1$ so that $E \cap \partial N$ is a simple closed curve in ∂E, and $E \cap \partial N^1$ is a simple closed curve in ∂E which bounds a disk in N^1. Using the techniques of Step 4, there is an annulus with orientable handles S properly embedded in $N - \operatorname{Int} N^1$ so that S has one boundary component in each of ∂N and ∂N^1, and so that $S \cap E$ is a spanning arc of both S and E. Thus the proof of Theorem 4 can be carried through to produce a map $f : N \to N$ with $f|\partial N =$ identity.

Question. Let G be a graph which is embedded in the interior of a 3-manifold M^3, and let N be a neighbourhood of G in M^3. Is there a monotone mapping f from M^3 onto itself with the following properties:

(1) $f|G$ is a homeomorphism,

(2) $f|M^3 - N$ is a homeomorphism,

(3) $f(M^3 - G) = M^3 - f(G)$,

(4) $f(G)$ is tame?

7. In this section, we give an alternative proof of Smythe's result [**11**] that any knot, link, or wedge of circles G, which is homologous to zero in an orientable 3-manifold M^3, is homologous to zero in a cube-with-handles $K \subset M^3$. We do not require, as Smythe does, that G be polyhedrally embedded. Smythe can obtain that if G bounds a singular surface of genus g in M^3, then it bounds a singular surface of the same genus in K. Our proof, however, gives no such bound on the genus of a surface S bounded by G in K.

COROLLARY 7. *Let G be a finite* 1-*complex topologically embedded in the interior of an orientable* 3-*manifold* M^3. *Suppose that each* 1-*simplex of G is oriented, and that G is a* 1-*cycle if each* 1-*simplex has coefficient* ± 1 *according to orientation. (Thus, for each vertex v of G, the number of edges of G pointing into v is the same as the number of edges pointing out from v.) If G, considered as a* 1-*cycle, is homologous to zero in* M^3, *then there is a compact* 3-*manifold* $K \subset \operatorname{Int} M^3$, *where each component of K is a cube-with-handles, such that G is homologous to zero in* Int K.

Remark. As special cases, G can be taken to be a simple closed curve, an oriented link, or an oriented wedge of simple closed curves.

Proof. We apply Theorem 1′ to each component of G to obtain a neighbourhood N of G where each component of N is a cube-with-handles. There is a collection $J_1, J_2, \ldots, J_m$ of oriented polyhedral simple closed curves in N so that $J = J_1 \cup J_2 \cup \ldots \cup J_m$ is homologous to G in N. Then J bounds a compact, orientable surface S (not necessarily connected). Recall from Theorem 1′ that for each vertex τ of some special decomposition P of G, there is a spanning disk D_τ of N. Using the techniques of Step 1 of § 4, we can assume that the surface S intersects the boundary of each disk D_τ exactly once. For each 1-element σ of P, there is a corresponding section $N_\sigma = T_\sigma \cup H_\sigma$ of N. Using the techniques of Step 1 again, we can assume that $S \cap (\partial N_\sigma \cap \partial N)$ contains no simple closed curve.

Let U be a regular neighbourhood of the surface S in $\operatorname{Int} M^3 - \operatorname{Int} N$. Then each component of U is a cube-with-handles. For each vertex τ of the special decomposition P, let V_τ be a regular neighbourhood of the disk D_τ in N which is so close to D_τ that $U \cap V_\tau$ is a disk. Then $U' = U \cup (\bigcup \{V_\tau : \tau \text{ is a vertex of } P\})$ is homeomorphic to U.

Each component of $N - \bigcup \{V_\tau : \tau \text{ is a vertex of } P\}$ is a cube-with-handles whose intersection with U' is a finite number of disks. Thus each component of $K = N \cup U$ is a cube-with-handles, and G is homologous to zero in Int K.

COROLLARY 8. *Let G be a finite* 1-*complex topologically embedded in the interior of a* 3-*manifold* M^3. *If G is inessential in* M^3, *then there is a compact* 3-*manifold* $K \subset \operatorname{Int} M^3$, *where each component of K is a cube-with-handles, so that G is inessential in* Int K.

Proof. Since G is an ANR, there is a neighbourhood N of G which is inessential in M^3. By Theorem 1′, N can be chosen so that it is compact and each component of N is a cube-with-handles. The required 3-manifold K can now be produced by the Corollary of [**11**] or the techniques of [**6**, § 2].

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CYCLIC MAPS FROM SUSPENSIONS TO SUSPENSIONS

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1. Introduction. In [7] Varadarajan defined the notion of a cyclic map $f : A \to X$. The collection of all homotopy classes of such cyclic maps forms the Gottlieb subset $G(A, X)$ of $[A, X]$. If $A = S^1$, this reduces to the group $G(X, x_0)$ of Gottlieb [**5**]. We show that a cyclic map f maps ΩA into the centre of ΩX in the sense of Ganea [**4**]. If A and X are both suspensions, we then show that if $f : A \to X$ maps ΩA into the centre of ΩX, then f is cyclic. Thus for maps from suspensions to suspensions, Varadarajan's cyclic maps are just those maps considered by Ganea. We also define $G(\Sigma A, \Sigma X)$ in terms of the generalized Whitehead product [**1**]. This gives the computations for $G(S^{n+k}, S^n)$ in terms of Whitehead products in $\pi_{2n+k}(S^n)$.

We work in the category of spaces with base points and having the homotopy type of countable CW-complexes. All maps and homotopies are with respect to base points. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.

Given spaces X and Y we denote the set of homotopy classes of maps from X to Y by $[X, Y]$. For any space X, we denote by $e : \Sigma\Omega X \to X$ the map whose adjoint is the identity map of ΩX and by $e' : X \to \Omega\Sigma X$ the map which is the adjoint of the identity map of ΩX, where Ω and Σ are the loop and suspension functors respectively.

2. We first state some definitions and results we shall need to prove our results. Let $f : A \to X$ be a map. We say that f is *cyclic* [7] if we can find a map $F : X \times A \to X$ such that $Fj = \nabla(1 \vee f)$ where $j : X \vee A \to X \times A$ is the inclusion of the wedge product into the cartesian product, and $\nabla : X \vee X \to X$ is the folding map. The set of all homotopy classes of such cyclic maps is the subset $G(A, X)$ of $[A, X]$.

If $f : A \to X$ is a map, then for every space Z, we have a homomorphism $(\Omega f)_\# : [Z, \Omega A] \to [Z, \Omega X]$. Let XbA be the flat product, that is, the fibre of the inclusion $j : X \vee A \to X \times A$. Then in [**4**], Ganea proved the following result.

THEOREM 1. *The following are equivalent:*
(i) $(\Omega f)_\#$ *maps* $[Z, \Omega A]$ *into the centre of the group* $[Z, \Omega X]$.
(ii) $\nabla(1 \vee f)\, i \simeq *$.

Any such map satisfying either of these conditions is referred to by Ganea as mapping ΩA into the centre of ΩX.

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THEOREM 2. *Let* $f : A \to X$ *be a cyclic map. Then* f *maps* ΩA *into the centre of* ΩX.

Proof. Let $q : X \vee A \to C$ be the cofibre of the inclusion $i : XbA \to X \vee A$. Then we have a map $r : C \to X \times A$ such that $rq = j : X \vee A \to X \times A$. Since f is cyclic, we have a map $F : X \times A \to X$ giving the following diagram with commutative triangles:

$$\begin{tikzcd}
& XbA \arrow[d, "i"] & \\
& X \vee A \arrow[r, "\nabla(1 \vee f)"] \arrow[dl, "q"'] \arrow[dd, "j"] & X \\
C \arrow[dr, "r"'] & & \\
& X \times A \arrow[uur, "F"'] &
\end{tikzcd}$$

Clearly $\nabla(1 \vee f)\, i \simeq *$. Hence f maps ΩA into the centre of ΩX.

THEOREM 3. *Let* $f : \Sigma A \to \Sigma X$ *map* $\Omega\Sigma A$ *into the centre of* $\Omega\Sigma X$. *Then* f *is cyclic.*

Proof. We have $\nabla(1 \vee f)\, i \simeq *$. By Lemma 5.1 of [**2**], it follows that $[e, fe] = 0$, where $e : \Sigma\Omega\,\Sigma X \to \Sigma X$, $fe : \Sigma\Omega\,\Sigma A \to \Sigma X$. Now in [**3**], Ganea showed that there is a map $\gamma : \Omega\Sigma X \wedge \Omega\Sigma X \to \Omega\Sigma X$ such that the composite $\gamma q : \Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X \wedge \Omega\Sigma X \to \Omega\Sigma X$ is the commutator of the two projections $\Omega\Sigma X \times \Omega\Sigma X \to \Omega\Sigma X$ in the group $[\Omega\Sigma X \times \Omega\Sigma X, \Omega\Sigma X]$. Then by Lemma 2.1 of [**2**], it follows that $[e, fe] = 0$ if and only if $\gamma\{\Omega e \wedge \Omega(fe)\} \simeq *$. If we use the same notation $e' : \Omega X \to \Omega\Sigma\,\Omega X$, $e' : \Omega A \to \Omega\Sigma\,\Omega A$ for the obvious embeddings, then we have $(\Omega e)e' \simeq 1_X$. Hence we have

$$\gamma(1_{\Omega\Sigma X} \wedge \Omega f) \simeq *.$$

Hence by the same Lemma 2.1 of [**2**], we have $[1_{\Sigma X}, f] = 0$. Now let $i_1 : \Sigma X \to \Sigma X \vee \Sigma A$, $i_2 : \Sigma A \to \Sigma X \vee \Sigma A$ be the usual inclusions. Then $\nabla(1_{\Sigma X} \vee f)[i_1, i_2] = [1_{\Sigma X}, f] = 0$. Since the cofibre of $[i_1, i_2] : \Sigma(X \wedge A) \to \Sigma X \vee \Sigma A$ is $(\Sigma X \vee \Sigma A) \cup C\Sigma(X \wedge A) \simeq \Sigma X \times \Sigma A$, it follows that we can find a map $F : \Sigma X \times \Sigma A \to \Sigma X$ such that $Fj = \nabla(1 \vee f)$ where $j : \Sigma X \vee \Sigma A \to \Sigma X \times \Sigma A$ is the inclusion. Hence f is cyclic.

Remark 1. In the course of the proof, we have shown that if $f : \Sigma A \to \Sigma X$ maps $\Omega\Sigma A$ into the centre of $\Omega\Sigma X$, then $[1_{\Sigma X}, f] = 0$. Conversely, it is obvious that if $[1_{\Sigma X}, f] = 0$, then f is cyclic. Thus we have the following corollary.

COROLLARY 3. *Let* $f : \Sigma A \to \Sigma X$. *Then the following are equivalent.*

(i) *f is cyclic.*
(ii) *f maps $\Omega\Sigma A$ into the centre of $\Omega\Sigma X$.*
(iii) $[1_{\Sigma X}, f] = 0$.

Remark 2. We can apply this result to spheres. Then we see that the computations of Varadarajan [**7**, Theorem 4.1] on $G(S^k, S^k)$ are just the well known results on the Whitehead product $[\iota, \iota]$. Further, the corollary could be applied to compute $G(S^{n+k}, S^n)$, for various k. The result depends on the computation of the Whitehead product on spheres. These have been extensively computed by Mahowald [**6**] and others.

We conclude by stating another result. We recall the following definition from [**7**], $P(\Sigma A, X) = \{\alpha \in [\Sigma A, X] \mid [\alpha, \beta] = 0$ for all $\beta \in [\Sigma^k A, X]$ and all $k \geqq 1\}$. Varadarajan proves that for all $k \geqq 1$, $G(S^k, S^k) = P(S^k, S^k)$. An obvious corollary of our results above is the following generalization,

THEOREM 4. $G(\Sigma X, \Sigma X) = P(\Sigma X, \Sigma X)$.

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THE EQUALITY $(A \cap B)^n = A^n \cap B^n$ FOR IDEALS

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1. Introduction. Let D be an integral domain with identity, and let R be a commutative ring. If n is a positive integer, R will be said to *have property* (n), $(n)'$, or $(n)''$ according as

property (n): For any $x, y \in R$, $(x, y)^n = (x^n, y^n)$.

property $(n)'$: For any $x \in R$ and any ideal A of R such that $x^n \in A^n$, it follows that $x \in A$.

property $(n)''$: For any ideals A, B of R, $(A \cap B)^n = A^n \cap B^n$.

J. Ohm introduced property (n) in [**7**] in connection with the question: If $n \geqq 2$ and if D has property (n), must D be a Prüfer domain? (The integral domain D with identity is a Prüfer domain if each nonzero finitely generated ideal of D is invertible; equivalently, D_P is a valuation ring for each proper prime ideal P of D.) Prior to Ohm's paper, it was known that if D has property (2) and if D is integrally closed, then D is Prüfer. In [**7**, Theorem 1.4], Ohm showed that D is Prüfer if D has property (2) and if 2 is a unit of D. Example 4.6 of [**7**] is a domain which has property (n) for all n, but which is not integrally closed, and hence not Prüfer.

In [**3**], Gilmer extended Ohm's investigation of property (n), and in the process he defined property $(n)'$. He showed that property $(n)'$ implies property (n) and that an integrally closed domain having property (n) for any $n > 1$, is Prüfer. Two examples in [**3**] show that a domain with property $(n)'$, for each positive n need not be Prüfer.

In this paper, we investigate the question:

Does property $(n)''$ imply that D is a Prüfer domain?

We show that property $(n)''$ implies both properties (n) and $(n)'$, and thus if D is integrally closed, it implies that D is a Prüfer domain. Example 3.4 shows, however, that property $(n)''$, for all n, is not strong enough to imply that the domain is integrally closed. Finally, we show in Example 3.7 that property $(n)''$ is not equivalent to either property (n) or $(n)'$. Our notation and terminology will be that of [**2**].

2. Property $(n)''$ and Prüfer domains. Throughout this section, D is an integral domain with identity, and n is an integer greater than 1.

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LEMMA 2.1. *Property* $(n)'' \Rightarrow$ *property* $(n)' \Rightarrow$ *property* (n).

Proof. Theorem 5.3 in [**3**] shows that property $(n)'$ implies property (n). To see that property $(n)''$ implies property $(n)'$, suppose that A is an ideal of D, $x \in D - \{0\}$, and $x^n \in A^n$. Then $(x)^n = (x)^n \cap A^n = [(x) \cap A]^n \subseteq (x)^{n-1}A$. Hence $(x) \subseteq A$ since $(x)^{n-1}$ is a cancellation ideal.

THEOREM 2.2. *Let D be an integrally closed domain. The following conditions are equivalent in D:*

(a) *D is a Prüfer domain.*

(b) *Property $(n)''$ holds in D.*

(c) *Property $(n)''$ for finitely generated ideals holds in D.*

Proof. (a) $\Rightarrow$ (b). It is clear that any valuation ring has property $(n)''$. Hence if D is Prüfer, and if $\{M_\lambda\}$ is the set of maximal ideals of D, then for any ideals A and B of D,

$$\begin{aligned}(A \cap B)^n &= \bigcap_\lambda (A \cap B)^n D_{M_\lambda} = \bigcap_\lambda [(A \cap B)D_{M_\lambda}]^n = \bigcap_\lambda (AD_{M_\lambda} \cap BD_{M_\lambda})^n \\ &= \bigcap_\lambda [(AD_{M_\lambda})^n \cap (BD_{M_\lambda})^n] \\ &= [\bigcap_\lambda A^n D_{M_\lambda}] \cap [\bigcap_\lambda B^n D_{M_\lambda}] = A^n \cap B^n.\end{aligned}$$

(b) $\Leftrightarrow$ (c). This is true in any commutative ring.

(b) $\Rightarrow$ (a). This follows from Lemma 2.1 above and from Theorem (20.3) in [**2**].

Remark 2.3. It is apparent from our proof of the implication (a) $\Rightarrow$ (b) that $[\bigcap_{i=1}^k A_i]^n = \bigcap_{i=1}^k A_i^n$ for any finite family $\{A_i\}_{i=1}^k$ of ideals of a Prüfer domain. The analogous equality for an arbitrary family of ideals of a Prüfer domain is not valid. For example, if $V = \mathbf{Q}[[X]]$ and if $M = XV$, where $\mathbf{Q}$ is the field of rational numbers, then if $\mathbf{Z}$ denotes the ring of integers, the domain $D = \mathbf{Z} + M$ is Prüfer [**2**, p. 561]; but if $A_i = p_i D$, where $p_1 < p_2 < \ldots$ is the sequence of positive prime integers, then for any positive integer n,

$$[\bigcap_{i=1}^{\infty} A_i]^n = M^n \subset \bigcap_{i=1}^{\infty} A_i^n = M.$$

At this point we detect a breakdown in the duality between the operations of addition and intersection on the set of ideals of a Prüfer domain, for it is true that

$$(\textstyle\sum_\lambda A_\lambda)^n = \sum_\lambda A_\lambda^n$$

for any family $\{A_\lambda\}$ of ideals of a Prüfer domain, and for any positive integer n.

Remark 2.4. A careful analysis of the proof of Lemma 2.1 and of Theorem 4.3 of [**3**] shows that the following generalization of the implication (c) $\Rightarrow$ (a) in Theorem 2.2 is valid.

(*) *If D is n-integrally closed and if* $(A \cap B)^n = A^n \cap B^n$ *for each pair* A, B *of ideals of D with a basis of two elements, then D is a Prüfer domain.* (If n is a positive integer and if J is an integral domain with identity with quotient field K, then J is said to be *n-integrally closed* [**3**] if J contains each element θ in K such that θ is a root of a monic polynomial $f(X) \in J[X]$ of degree n.)

Result (*) is of some interest because of its connection with one of the more important open questions concerning Prüfer domains, namely: Does every finitely generated ideal of a Prüfer domain have a basis of two elements? [**5**].

Remark 2.5. The concept of a Prüfer domain has been extended to commutative rings with zero divisors, thereby obtaining Prüfer rings. M. Griffin's paper [**6**] contains much of what is known about Prüfer rings. Using the results of [**6**], it is straightforward to prove the following generalization of Theorem 2.2:

Let R be an integrally closed ring with few zero divisors, and let n be an integer greater than one. The following conditions are equivalent in R:

(a) *R is a Prüfer ring.*

(b) *If A and B are regular ideals of R, then* $(A \cap B)^n = A^n \cap B^n$.

(c) *If A and B are finitely generated regular ideals of R, then* $(A \cap B)^n = A^n \cap B^n$.

The question arises as to the relationship between conditions (a), (b), and (c) if the hypothesis "R has few zero divisors" is dropped. In partial answer to this question, we can prove (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). An examination of our proof of Lemma 2.1, together with results of [**3**], show that (c) implies the following condition (c)′:

(c)′: If $\{r_i\}_{i=1}^n$ is a finite set of regular elements of R, then $(r_1, \ldots, r_n)$ is invertible.

An example in [**4**] shows that an integrally closed ring in which (c)′ holds need not be a Prüfer ring, but we have no example to show that (c) does not imply (a).

3. Examples.

In this section, we present a class of domains with property $(n)''$ for every positive n, but which are not integrally closed.

Let V be a valuation ring of the form $K + M$, where K is a field and M is the maximal ideal of V, and let v be a valuation associated with V. Let k be a proper subfield of K, and set $D = k + M$. In order to present Example 3.4, we shall need a characterization of the finitely generated ideals of D and of the powers of such ideals.

LEMMA 3.1 (Gilmer [**3**]). *If* $x \in D - \{0\}$, xD *contains each element* y *of* V *such that* $v(y) > v(x)$. *If A is a finitely generated ideal of D, say* $A = (a_1, \ldots, a_n)$, *and if* $t = \min\{v(a_i) | 1 \leq i \leq n\}$, *then for any element a of A such that* $v(a) = t$, *A has a basis* $\{a, k_2a, \ldots, k_ma\}$ *for some* $k_2, \ldots, k_m \in K - k$. *Moreover,*

$A = Wa + C$, where W is the k-subspace of K spanned by $\{1, k_2, \dots, k_m\}$ and $C = \{y \in V | v(y) > t\}$.

LEMMA 3.2. *Let $A = Wa + C$ be a finitely generated ideal of D as given in Lemma 3.1. Then $A^n = W^n a^n + C_1$, where $C_1 = \{y \in V | v(y) > nv(a)\}$. If B is a finitely generated ideal of D of the form $W_1 b + C$, where $v(b) = v(a)$ and W_1 is a finite-dimensional k-subspace of K, then*

$$(A \cap B)^n = (Wa \cap W_1 b)^n + C_1.$$

Proof. Any element $x \in A^n$ is a finite sum of elements $a_1 a_2 \dots a_n$, $a_i \in A$. Writing $a_i = w_i a + c_i$, $w_i \in W$, $c_i \in C$, we obtain $a_1 a_2 \dots a_n = w_1 \dots w_n a^n + w_1 \dots w_{n-1} a^{n-1} c_n + \dots + c_1 c_2 \dots c_n$. Since each term in this expression except the first has v-value greater than $nv(a)$, $a_1 \dots a_n \in W^n a^n + C_1$. Hence $x \in W^n a^n + C_1$.

Conversely, $Wa \subseteq A$ implies $W^n a^n \subseteq A^n$. Also if $y \in C_1$, then $z(y) > nv(a) = v(a^n)$ implies that $y \in a^n D \subseteq A^n$. It then follows that $W^n a^n + C_1 \subseteq A^n$, and so equality holds.

The proof that $(A \cap B)^n = (Wa \cap W_1 b)^n + C_1$ follows similarly.

If R is a subring of the commutative ring S, then R is said to have *property (n) with respect to S* if for each $\xi \in S$, there exist $a_i, b_i \in R$ such that $\xi^i = a_i \xi^n + b_i$, $i = 1, \dots, n-1$. We are interested in the case where R and S are fields.

THEOREM 3.3. *Let V be a valuation ring of the form $K + M$, K a field, M the maximal ideal of V, and let v be a valuation associated with V. Suppose that k is a proper subfield of K such that k has property (n) with respect to K for some positive integer n. Then $D = k + M$ has property $(n)''$.*

Proof. We remark that Ohm [**7**] observed that A has property (n), and Gilmer [**3**] showed that A has property $(n)'$.

Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be finitely generated ideals of D, let $t_1 = \min \{v(a_i) | 1 \leqq i \leqq n\}$ and $t_2 = \min \{v(b_i) | 1 \leqq i \leqq m\}$. If $t_1 > t_2 = v(b_j)$, then by Lemma 3.1, each $a_i \in b_j D$, so that $A \subseteq B$, and the result is clear. Thus we may assume that $t_1 = t_2$, $A = W_1 a + C$ and $B = W_2 b + C$, $a \in A$, $b \in B$, $v(a) = v(b) = t_1$, $C = \{y \in V | v(y) > t_1\}$, and W_1, W_2 are finite-dimensional k-subspaces of K.

In [**1**], J. W. Brewer showed that for k to have property (n) with respect to K, it is necessary that $[K : k] = 2$. It follows that we have the following three cases to consider:

1. $A = Ka + C$, $B = Kb + C$.
2. $A = ka + C$, $B = Kb + C$.
3. $A = ka + C$, $B = kb + C$.

Since $v(a) = v(b)$, there exists $\gamma \in K - \{0\}$ and $m \in M$ such that $b/a = \gamma + m$. Since $am \in C$, there is no loss of generality in assuming that $m = 0$

(that is, γa is in B and is an element of B of minimal value). Using this relationship and Lemma 3.2, the three cases become

1. $A = Ka + C = K\gamma a + C = Kb + C = B.$
2. $A = ka + C \subseteq Ka + C = K\gamma a + C = B.$
3. $(A \cap B)^n = (ka \cap k\gamma a)^n + C_1 = (k \cap k\gamma)^n a^n + C_1.$
   $A^n \cap B^n = [k^n a^n \cap k^n(\gamma a)^n] + C_1 = (k \cap k\gamma^n)a^n + C_1.$

Now the containment $(k \cap k\gamma)^n \subseteq k \cap k\gamma^n$ always holds, and since k, $k\gamma$ and $k\gamma^n$ are one-dimensional k-subspaces of K, it follows that, for $1 \leqq i \leqq n$, $k \cap k\gamma^i$ is either k or (0), depending upon whether γ^i is, or is not, in k. Since k has property (n) with respect to K, it follows from Lemma 5.5 of [**3**] that $\gamma \in k$ if and only if $\gamma^n \in k$. Thus $(k \cap k\gamma)^n = (k \cap k\gamma^n)$, and hence $(A \cap B)^n = A^n \cap B^n$.

Example 3.4. In [**7**], Ohm constructed fields k, K, with k a proper subfield of K, such that k has property (n) with respect to K for each positive integer n. If M is the maximal ideal of the valuation ring $K[[X]]$, then the domain $D = k + M$ has property $(n)''$ for each n. Since $K[[X]]$ in the integral closure of D, D is not Prüfer.

Our next example shows that property $(n)''$ is indeed stronger than properties (n) and $(n)'$. Suppose that V_1 and V_2 are rank one discrete valuation rings having a common quotient field L, that K is a common subfield of V_1 and V_2, and that $V_i = K + M_i$, where M_i is the maximal ideal of V_i. We are interested in the domain $D = K + (M_1 \cap M_2)$. If v_i is a valuation associated with V_i, then by the approximation theorem for independent valuations, there exist $a, b \in L$ such that $v_1(a) = v_2(b) = 1$, $v_1(b) = v_2(a) = 0$. Using this notation, we have

LEMMA 3.5. *The domain D is local. In particular, if A is a nonprincipal ideal of D, then there exist positive integers n, m such that $A = (a^n b^m, a^{n+1}b^m)$. Moreover,*

$$(a^n b^m, a^{n+1}b^m) = \{d \in D \mid v_1(d) \geqq n,\ v_2(d) \geqq m\}.$$

Proof. We show first that if $t \geqq n$, $s \geqq m$, and if $v_1(x) = v_2(x) = 0$, then there exist $\xi_1, \xi_2 \in K$ and $z \in M_1 \cap M_2$ such that

$$(\#) \qquad a^t b^s x = (\xi_1 + z)a^n b^m + \xi_2 a^{n+1} b^m.$$

Suppose that $a^{t-n}b^{s-m}\, x \equiv \mu_i(M_i)$, and that $a \equiv \eta(M_2)$, where $\mu_i, \eta \in K$, $\eta \neq 0$. Then $\xi_1 = \mu_1$, $\xi_2 = \eta^{-1}(\mu_2 - \mu_1)$ is the unique solution in K of the system of equations

$$\mu_1 = X,$$
$$\mu_2 = X + \eta Y.$$

It follows that $a^{t-n}\, b^{s-m}\, x - \xi_1 - a\xi_2 = z \in M_1 \cap M_2$. Hence $(\#)$ follows, and from this we have $(a^n b^m, a^{n+1}b^m) = \{d \in D \mid v_1(d) \geqq n,\ v_2(d) \geqq m\}$.

Now let A be a nonprincipal ideal of D. Let $n = \min \{v_1(x) | x \in A\}$, let $m = \min \{v_2(x) | x \in A \text{ and } v_1(x) = n\}$, and let y be an element in A such that $v_1(y) = n$, $v_2(y) = m$. Write $y = a^n b^m u$, where $v_1(u) = v_2(u) = 0$. We choose $x \in A - (a^n b^m u)D$. It is clear that $v_1(x) \geqq n$. We show that $v_2(x) \geqq m$. If $v_1(x) = n$, it is clear that $v_2(x) \geqq m$; and if $v_1(x) > n$, then $x + a^n b^m u \in A$, $v_1(x + a^n b^m u) = n$, and hence $v_2(x + a^n b^m u) \geqq m$, so that $v_2(x) \geqq m$. It then follows from (#) that there exist $\xi_1, \xi_2 \in K$ and $z \in M_1 \cap M_2$ such that $xu^{-1} = (\xi_1 + z)a^n b^m + \xi_2 a^{n+1} b^m$. Therefore, $x = (\xi_1 + z)a^n b^m u + \xi_2 u a^{n+1} b^m = (\xi_1 + z)a^n b^m u + \xi_2 a^{n+1} b^m (\mu + hb)$, where $u = \mu(M_2)$ and where $h \in V_2$. It then follows that $x = (\xi_1 + z + \xi_2 hab)a^n b^m + \mu\xi_2 a^{n+1} b^m$, where $\xi_2 hab \in M_1 \cap M_2$. Since $x \notin a^n b^m uD$, $\mu\xi_2 \neq 0$ and thus $a^{n+1} b^m \in (x, a^n b^m u) \subseteq A$. Hence $(a^n b^m, a^{n+1} b^m) \subseteq A \subseteq (a^n b^m, a^{n+1} b^m)$, and equality follows.

THEOREM 3.6. *Let n be an integer greater than one. The domain $D = K + (M_1 \cap M_2)$ does not have property $(n)''$. D has property $(n)'$ if and only if the mapping $x \rightarrow x^n$ of K into K is one-to-one.*

Proof. Let $A = (ab)D$ and $B = (a^2 b)D$. Then $(A \cap B)^n = (a^{3n} b^{2n}, a^{3n+1} b^{2n})$ while $A^n \cap B^n = (a^{2n+1} b^{n+1}, a^{2n+2} b^{n+1})$, so that $(A \cap B)^n \subset A^n \cap B^n$.

Theorem 7.1 of [**3**] shows that D has property (n) if and only if the mapping $x \rightarrow x^n$ of K is one-to-one. Hence it suffices to show that if D has property (n), then it also has property $(n)'$.

Now if J is any domain with property (n), then J has property $(n)'$ with respect to principal ideals. That is, if $x^n \in (y)^n$, then $x \in (y)$, for since J has property (n), $xy^{n-1} \in (x^n, y^n) = (y^n)$. Thus we need only consider the nonprincipal ideals of D.

If A is a nonprincipal ideal of D, then it follows from Lemma 3.5 that $A = a^n b^m (V_1 \cap V_2)$ is an ideal of $V_1 \cap V_2$, and hence A is an intersection of valuation ideals. Lemma 5.1 of [**3**] shows that if the ideal A of the domain D is an intersection of valuation ideals of D, and if $x \in D$ is such that $x^n \in A^n$, then $x \in A$. Thus the proof is complete.

Example 3.7. The prime field π_2 with two elements has the property that $x \rightarrow x^n$ is one-to-one for any positive integer n. Let $V_1 = (\pi_2[X])_{(X)} = \pi_2 + M_1$, and let $V_2 = (\pi_2[X])_{(X+1)} = \pi_2 + M_2$. Then if $D = \pi_2 + (M_1 \cap M_2)$, we obtain an example of a domain having property $(n)'$ for each positive integer, but having property $(n)''$ for no $n > 1$.

Remark 3.8. For a positive integer $n > 1$, there are essentially two different methods of obtaining domains with property (n) which are not integrally closed. One is the $k + M$ construction of our Theorem 3.3. As we have previously remarked, Gilmer in [**3**] showed that domains constructed in this way have property $(n)'$. The second way of obtaining non-integrally closed domains with property (n) is the $K + (M_1 \cap M_2)$-construction of our Theorem 3.6. As we have shown in the proof of Theorem 3.6, property (n) in such a domain is equivalent to property $(n)'$. It follows that no example has been pointed out in the literature to show that (n) does not imply $(n)'$. In fact, it is conceivable that the properties $(n)'$, (n), and $(n)^*$ of [3] are equivalent.

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ON CONFORMALLY FLAT SPACES WITH COMMUTING CURVATURE AND RICCI TRANSFORMATIONS

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Let (M, g) be a C^∞ Riemannian manifold and A be the field of symmetric endomorphisms corresponding to the Ricci tensor S; that is,

$$S(X, Y) = g(AX, Y).$$

We consider a condition weaker than the requirement that A be parallel ($\nabla A = 0$), namely, that the "second exterior covariant derivative" vanish ($\nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A = 0$), which by the classical interchange formula reduces to the property

$$(P) \qquad R(X, Y) \circ A = A \circ R(X, Y),$$

where $R(X, Y)$ is the curvature transformation determined by the vector fields X and Y.

The property (P) is equivalent to

$$(Q) \qquad R(AX, X) = 0.$$

To see this we observe first that a skew symmetric and a symmetric endomorphism commute if and only if their product is skew symmetric. Thus we have

$$\begin{aligned} (P) &\Leftrightarrow R(Z, W)A \text{ is skew symmetric} \\ &\Leftrightarrow g(R(Z, W)AX, X) = 0 \\ &\Leftrightarrow g(R(AX, X)Z, W) = 0 \\ &\Leftrightarrow (Q). \end{aligned}$$

Let M be a connected conformally flat manifold of dimension $n, n \geqq 3$. Then the Ricci endomorphisms determine the curvature according to the formula

$$(1) \qquad R(X, Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y,$$

where $r = \operatorname{trace} A$ and $X \wedge Y$ denotes the endomorphism

$$Z \rightarrow g(Y, Z)X - g(X, Z)Y.$$

In this paper the connected conformally spaces satisfying (P) are classified.

LEMMA 1. *Let M be an n-dimensional conformally flat space satisfying (P).*

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Then

$$A^2 - \frac{r}{n-1} A = \rho I, \tag{2}$$

where ρ is a C^∞ function on M and I is the identity field.

Proof. Setting $Y = AX$ in (1) and then applying (Q) gives

$$BX \wedge X = 0, \tag{3}$$

where

$$B = A^2 - \frac{r}{n-1} A.$$

Since (3) may be interpreted as an exterior product, we conclude that every X is an eigenvector of B, so $B = \rho I$ for some scalar field ρ.

LEMMA 2. *Under the conditions in Lemma 1, A has at most the two eigenvalues*

$$\frac{r \pm [r^2 + 4(n-1)\rho]^{\frac{1}{2}}}{2(n-1)}.$$

Let M' be the open subset of M on which $r^2 + 4(n-1)\rho \neq 0$. Then the eigenspaces of A form smooth complementary orthogonal distributions on each connected component of M'.

The eigenvalues are the roots of

$$\mu^2 - \frac{r}{n-1}\mu - \rho = 0;$$

the rest is also routine.

Let us fix notation as follows: The eigenvalues of A are μ_1 and μ_2. They are defined and continuous on all of M and distinct on M'. The eigenspaces on M' are D_1 and D_2, of dimensions k and $n - k$. We shall use adapted orthonormal frames and coframes $\{X_a, X_\alpha\}$ and $\{\omega_a, \omega_\alpha\}$, $a, b = 1, \dots, k$ and $\alpha, \beta = k+1, \dots, n$; moreover, $i, j = 1, \dots, n$. The corresponding connection and curvature forms are ω_{ab}, etc. and Ω_{ab}, etc.

LEMMA 3. *Let $K = (\mu_1 - \mu_2)/(n-2)$. On D_1 the sectional curvature is K, on D_2 it is $-K$, and on mixed sections it vanishes; that is,*

$$\begin{aligned} \Omega_{ab} &= K\omega_a \wedge \omega_b, \\ \Omega_{\alpha\beta} &= -K\omega_\alpha \wedge \omega_\beta, \\ \Omega_{a\beta} &= 0. \end{aligned}$$

Proof. Noting that $r/(n-1) = \mu_1 + \mu_2$, formula (1) becomes

$$R(X, Y) = \frac{1}{n-2}\{AX \wedge Y + X \wedge AY - (\mu_1 + \mu_2) X \wedge Y\}.$$

The rest follows by taking orthonormal eigenvectors for X and Y.

Note that M' is just the set on which $K \neq 0$.

THEOREM. *Let M be an n-dimensional connected conformally flat space satisfying (P), $n \geqq 3$. Then M is one of four types:*

(a) *M is flat (M' empty).*

In the remaining cases $M = M'$; that is, M is either flat everywhere or has no flat points; moreover, k is constant.

(b) *M has constant curvature ($k = 0$ or n).*

(c) *M is locally the Riemannian product of a k-dimensional space of constant curvature K and an $(n-k)$-dimensional space of constant curvature $-K (1 \leqq k \leqq n-1)$.*

(d) *There is an open C^∞ map $t : M \to \mathbf{R}^+$ (positive reals) such that $K = K_0/t^2$ for some constant K_0. The map t is a Riemannian submersion having fibres which are totally umbilical hypersurfaces of constant (intrinsic) curvature $(1 + K_0)/t^2$ ($k = 1$ or $n - 1$).*

Proof. Define the vector valued 1-form $F = F^i \otimes X_i$ by

$$F^i = A^i{}_j\omega^j - \frac{r}{2(n-1)}\omega^i,$$

where $A^i{}_j$ are the components of A. (The summation convention is employed here and in the sequel.) The X_i and ω^i are any local vector field basis and the dual basis of 1-forms, respectively. If $\omega^i{}_j$ are the connection forms for this basis we define the exterior covariant derivative DF of F as the vector-valued 2-form $(DF)^i \otimes X_i$, where

$$(DF)^i = dF^i + \omega^i{}_j \wedge F^j.$$

It is easily checked that DF is independent of the choice of basis. Using the first structural equation viz., $d\omega^i = -\omega^i{}_j \wedge \omega^j$, and the coefficients Γ^i_{kj} of $\omega^i{}_j (\omega^i{}_j = \Gamma^i_{kj}\omega^k)$, we obtain

$$\begin{aligned}(DF)^i &= \left(X_k A^i{}_j + A^i{}_h\Gamma^h_{kj} + A^h{}_j\Gamma^i_{kh} - \frac{1}{2(n-1)}\delta^i{}_j X_k r\right)\omega^k \wedge \omega^j \\ &= \left(\nabla_k A^i{}_j - \frac{1}{2(n-1)}\delta^i{}_j \nabla_k r\right)\omega^k \wedge \omega^j,\end{aligned}$$

where $\delta^i{}_j$ is the Kronecker delta. As a tensor, this has the components

$$(n-2)C^i{}_{jk} = \nabla_k A^i{}_j - \nabla_j A^i{}_k - \frac{1}{2(n-1)}(\delta^i{}_j \nabla_k r - \delta^i{}_k \nabla_j r),$$

where $C^i{}_{jk}$ is Weyl's 3-index tensor. For a conformally flat space it is known that $C^i{}_{jk} = 0$. We use this by calculating DF in terms of an orthonormal basis adapted to the distributions D_i. In particular we can lower all superscripts.

Thus,

$$\begin{aligned} F_a &= A_{ai}\omega_i - \frac{r}{2(n-1)}\,\omega_a \\ &= \mu_1\omega_a - \tfrac{1}{2}(\mu_1+\mu_2)\omega_a \\ &= L\omega_a, \end{aligned}$$

where $L = (n-2)K/2$, and

$$\begin{aligned} F_\alpha &= A_{\alpha i}\omega_i - \frac{r}{2(n-1)}\,\omega_\alpha \\ &= \mu_2\omega_\alpha - \tfrac{1}{2}(\mu_1+\mu_2)\omega_\alpha \\ &= -L\omega_\alpha, \end{aligned}$$

from which

$$\begin{aligned} dF_a &= dL \wedge \omega_a + Ld\omega_a + \omega_{ab} \wedge L\omega_b - \omega_{a\beta} \wedge L\omega_\beta \\ &= dL \wedge \omega_a - L\omega_{ai} \wedge \omega_i + L(\omega_{ab} \wedge \omega_b - \omega_{a\beta} \wedge \omega_\beta) \\ &= dL \wedge \omega_a - 2L\omega_{a\beta} \wedge \omega_\beta \\ &= 0, \end{aligned} \tag{4}$$

$$\begin{aligned} dF_\alpha &= -dL \wedge \omega_\alpha + 2L\omega_{\alpha b} \wedge \omega_b \\ &= 0. \end{aligned} \tag{5}$$

When $k = n$, K is constant and $M' = M$ follows immediately from Schur's theorem (or (4)).

Otherwise, by Cartan's lemma, (4) says that for each a, dL and the $\omega_{a\beta}$ are dependent at most on ω_a and the ω_α and (5) says that the same forms are dependent at most on ω_α and the ω_b. Thus if $2 \leqq k \leqq n-2$ we can make two choices of α for each a and vice-versa, showing that $dL = 0$ and $\omega_{a\beta} = 0$. Consequently, L and $K = 2L/(n-2)$ are constant and D_1 and D_2 are parallel (in particular, completely integrable).

When $k = 1$ we still have by (5) that dL and $\omega_{\alpha 1}$ are dependent at most on ω_α and ω_1. Making two choices of α, we get $dL = H\omega_1$ for some C^∞ function H. Then, (4) reduces to $\omega_{1\beta} \wedge \omega_\beta = 0$, so the $\omega_{1\beta}$ cannot depend on ω_1. Hence $\omega_{1\alpha} = C_\alpha\omega_\alpha$ (α not summed) for some scalar field C_α. But then by (5) again

$$-H\omega_1 \wedge \omega_\alpha + 2L(-C_\alpha\omega_\alpha) \wedge \omega_1 = 0;$$

that is, $C = C_\alpha = H/2L$ is the same for all α. The geometrical interpretation of the relation $\omega_{1\alpha} = C\omega_\alpha$ is that D_2 (the distribution annihilated by ω_1) is completely integrable and has totally umbilical leaves. In fact, $d\omega_1 = -\omega_{1\alpha} \wedge \omega_\alpha = 0$, so locally ω_1 has a primitive u; that is, $du = \omega_1$.

A differential equation for C may be obtained from the fact that the curvature of the section $X_1 \wedge X_\alpha$ vanishes:

$$\begin{aligned}\Omega_{1\alpha} &= d\omega_{1\alpha} + \omega_{1\beta} \wedge \omega_{\beta\alpha} \\ &= dC \wedge \omega_\alpha + C d\omega_\alpha + \omega_{1\beta} \wedge \omega_{\beta\alpha} \\ &= dC \wedge \omega_\alpha - C\omega_{\alpha 1} \wedge \omega_1 + C\omega_\beta \wedge \omega_{\beta\alpha} \\ &= \left(\frac{dC}{du} - C^2\right) \omega_1 \wedge \omega_\alpha.\end{aligned}$$

Therefore,

$$\frac{dC}{du} - C^2 = 0.$$

Solving this, we obtain either $C = 0$ or

$$C = -\frac{1}{u - u_0} = -\frac{1}{t},$$

where u_0 is a constant and hence t is another primitive for ω_1. The signs of C and ω_1 can be changed, if necessary, so as to make $t > 0$.

If $C = 0$, then it must be so on connected sets. Hence $H = dL/du = 2LC = 0$ and L, and hence K, are constant. Moreover, $C = 0$ says D_1 and D_2 are parallel so we are back in case (c).

If $C \neq 0$, then we solve $H = 2LC$ for L, obtaining $L = L_0/t^2$, and hence $K = K_0/t^2$ for constants L_0 and K_0. Thus, $t = (K_0/K)^{\frac{1}{2}}$ is a primitive for ω_1 in each component of M'. We don't know yet whether there is only one component, so K_0 might have several values. As a map $t : M' \to \mathbf{R}^+$, t is clearly a Riemannian submersion whose fibres are the leaves of D_2. As such it is distance-non-increasing. Now suppose that $M' \neq M$. Let γ be a curve entirely in M' except for the last point $\gamma(1) \in M - M'$. The length of $t\gamma$ is at most that of γ and is therefore bounded. Hence $t\gamma(1) = \lim_{s\to 1^-} t\gamma(s)$ exists and is not ∞. It cannot be 0 either, for then there would be a sequence of plane sections converging to a section at $\gamma(1)$ and having curvatures diverging to $\lim_{t\to 0} K_0/t^2$. A similar difficulty is presented at any other finite limit for $t\gamma(1)$, since we would then have curvatures converging to nonzero values contradicting the fact that $M - M'$ is flat. Hence, $M = M'$.

To complete the proof we calculate the intrinsic curvature of the leaves of D_2. The connection forms $\omega_{\alpha\beta}$, restricted to a leaf, become the connection forms of the leaf. Thus, denoting the curvature forms of a leaf by $\Phi_{\alpha\beta}$, the second structural equation for a leaf is

$$\begin{aligned}d\omega_{\alpha\beta} &= -\omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + \Phi_{\alpha\beta} \\ &= -\omega_{\alpha 1} \wedge \omega_{1\beta} + \Omega_{\alpha\beta} \\ &= -\omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} + (C^2 + K)\omega_\alpha \wedge \omega_B.\end{aligned}$$

Evidently the curvature forms of the leaf are

$$\Phi_{\alpha\beta} = (C^2 + K)\omega_\alpha \wedge \omega_\beta,$$

so the curvature of the leaves of D_2 is $(1 + K_0)/t^2$.

Remark. If M is complete, then the case (d) cannot occur, since the base of a complete Riemannian submersion must be complete.

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ON DIRAC'S GENERALIZATION OF BROOKS' THEOREM

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1. Introduction. It is easy to verify that any connected graph G with maximum degree s has chromatic number $\chi(G) \leqq 1 + s$. In [**1**], R. L. Brooks proved that $\chi(G) \leqq s$, unless $s = 2$ and G is an odd cycle or $s > 2$ and G is the complete graph K_{s+1}. This was the first significant theorem connecting the structure of a graph with its chromatic number. For $s \geqq 4$, Brooks' theorem says that every connected s-chromatic graph other than K_s contains a vertex of degree $> s - 1$. An equivalent formulation can be given in terms of s-critical graphs. A graph G is said to be *s-critical* if $\chi(G) = s$, but every proper subgraph has chromatic number less than s. Each s-critical graph has minimum degree $\geqq s - 1$. We can now restate Brooks' theorem: if an s-critical graph, $s \geqq 4$, is not K_s and has p vertices and q edges, then $2q \geqq (s - 1)p + 1$. Dirac [**2**] significantly generalized the theorem of Brooks by showing that $2q \geqq (s - 1)p + s - 3$ and that this result is best possible. Dirac's theorem has several important applications. For example, Dirac [**3**] used his result to show that if a graph G with genus $n \geqq 1$ has

$$\chi(G) = \left[\frac{7 + (1 + 48n)^{\frac{1}{2}}}{2}\right] = H(n),$$

then G contains $K_{H(n)}$ as a subgraph. The object of this note is to present a new proof of Dirac's theorem.

2. Dirac's theorem. In this section we state and prove Dirac's theorem. Although our proof is not particularly short, it is considerably shorter than the original one [**2**]. The first part of the proof below was suggested by Melnikov and Vizing's [**4**] elegant new proof of Brooks' theorem.

THEOREM. *If G is an s-critical graph, $s \geqq 4$, which is not complete, then*

$$2q \geqq (s - 1)p + s - 3.$$

Proof. Suppose the theorem is false. Then there exists an s-critical, $s \geqq 4$, graph $H \neq K_s$, with $2q \leqq (s - 1)p + s - 4$. Since H is s-critical, $2q \geqq (s - 1)p$, so that at most $s - 4$ vertices of H have degree $\geqq s$. Let v be a vertex of degree $s - 1$ and let $H' = H - v$. The graph H' has $\chi(H') = s - 1$. In each $(s - 1)$-coloring of H' the vertices $v_1, v_2, \ldots, v_{s-1}$ adjacent to v must necessarily be colored in different colors, say $1, 2, \ldots, s - 1$, respectively.

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Also each v_i must be adjacent to a vertex colored $j \neq i$, $1 \leqq j \leqq s - 1$.

Assume the graph H' has been colored as above, then:

1. *Vertices v_i and v_j $(i, j = 1, \ldots, s - 1, i \neq j)$ are in the same component C_{ij} of the subgraph induced by vertices colored i and j.* Otherwise the interchanging of colors i and j in the component containing v_i would give an $(s - 1)$-coloring of H' in which v_i and v_j have the same color. A v_iv_j-path in C_{ij} will be denoted by P_{ij}.

We let N denote the number of colors which are assigned to vertices having degree $\geqq s$ in H. Also let $n = s - 1 - N$ and assume that $1, 2, \ldots, n$ are the colors assigned only to vertices having degree $s - 1$ in H. We note that $n \geqq 3$, since $0 \leqq N \leqq s - 4$. Also, for $1 \leqq i \leqq$ n, v_i is adjacent to exactly one vertex of each color $j \neq i$, $1 \leqq j \leqq s - 1$.

2. *The component C_{ij} $(i, j = 1, 2, \ldots, n; i \neq j)$ is a path.* All the vertices of C_{ij} have degree $s - 1$ in H. It follows from above that the vertices v_i and v_j have degree 1 in C_{ij}. All other vertices must have degree 2; otherwise in moving from v_i to v_j along P_{ij}, the first vertex u having degree > 2 in C_{ij} would have degree $\geqq s$ in H. If u had degree $s - 1$, then it could be recolored in a color different from i and j so that v_i and v_j would lie in different components, in contradiction to 1.

3. *The paths P_{ij} and P_{ik} $(i, j, k = 1, 2, \ldots, n; i \neq j \neq k \neq i)$ have no common vertex except v_i.* If they had a common vertex different from v_i of degree $s - 1$ in H, then it could be recolored in a color different from i, j, k so that v_i and v_j would not be joined by a P_{ij} path.

4. *If $i, j = 1, \ldots, n; i \neq j$, then v_i and v_j are adjacent.* Assume without loss of generality that v_1 and v_2 are not adjacent. Then the path P_{12} contains a vertex y adjacent with v_1 and different from v_2. Interchange the colors along P_{13}. After this change, the new paths P_{12} and P_{23} will each have the common vertex y, in contradiction to 3.

Since $H \neq K_s$, there exist non-adjacent vertices v_α and v_β $(\alpha, \beta = 1, 2, \ldots, s - 1; \alpha \neq \beta)$. At least one color class determined by the $s - 3$ colors different from α and β has only one point adjacent to v_α and one point adjacent to v_β. Otherwise, $\deg v_\alpha + \deg v_\beta \geqq 2(s - 1) + s - 3$ and hence $2q \geqq (s - 1)p + s - 3$. Let the colors that meet this condition be $1', 2', \ldots, t'$.

5. $\{1', 2', \ldots, t'\} \cap \{1, 2, \ldots, n\} \neq \phi$. Otherwise each color $1', \ldots, t'$ is associated with a vertex of degree $\geqq s$. Also, the $s - 3 - t$ colors different from $1', \ldots, t', \alpha, \beta$, each have two vertices adjacent to v_α or two vertices adjacent to v_β. Again we obtain $2q \geqq (s - 1)p + s - 3$. We can assume that $1 \in \{1', \ldots, t'\} \cap \{1, \ldots, n\}$.

6. *v_1 is adjacent to at most one of the vertices v_α and v_β.* If v_1 is adjacent to both v_α and v_β, then $C_{1\alpha} = (v_1, v_\alpha)$ and $C_{1\beta} = (v_1, v_\beta)$. If we now interchange the colors along $C_{1\alpha}$, then we obtain a coloring in which v_β is not adjacent to any vertex colored 1. We assume that v_1 and v_β are not adjacent.

Using arguments like that used for statement 2, it is easy to verify that for each i, $2 \leqq i \leqq n$, such that $C_{\beta i}$ is not a path, there is a vertex colored β in

$C_{\beta i}$ having degree $\geqq s$. These vertices are not necessarily distinct, however, if a vertex occurs k times, then its degree is at least $s - 1 + k$.

7. *There exists at least one* i, $2 \leqq i \leqq n$, *such that* $C_{\beta i}$ *is a path*. Otherwise, by the above remarks, if there are p' vertices colored β, then the sum of the degrees of these vertices is at least $(s - 1)p' + n - 1$. We also have at least $N - 1$ vertices having degree $\geqq s$ which are not colored β. Again, we have $2q \geqq (s - 1)p + s - 3$.

We let $2, 3, \ldots, m$, denote the colors which satisfy 7. By 6, there is a path $P_{1\beta}$ which contains a vertex u adjacent with v_1 and different from v_β.

8. *The vertex* u *does not belong to each* $C_{\beta i}$, $2 \leqq i \leqq m$. If it did, then u would have degree $\geqq 2m + (s - 1 - (m + 1)) = (s - 1) + (m - 1)$. By the remarks preceding 7, the sum of the degrees of the p' vertices colored β is at least $(s - 1)p' + (n - m) + (m - 1) = (s - 1)p' + n - 1$. There are also $(s - 1) - (n + 1)$ distinct vertices having degree $\geqq s$ which are colored with colors from the set $\{n + 1, n + 2, \ldots, s - 1\} - \{\beta\}$. Therefore, $2q \geqq (s - 1)p + s - 3$.

To produce a final contradiction, we let 2 be a color satisfying statements 7 and 8. Then u is not on $P_{\beta 2}$. By 4, $P_{12} = (v_1, v_2)$ so that if we interchange colors along $P_{\beta 2}$, then we obtain a coloring having the property that v_1 is not adjacent to any vertex colored 2. This is impossible and hence the assumption that $2q \leqq (s - 1)p + s - 4$ is false.

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ASSOCIATED PRIME DIVISORS IN THE SENSE OF KRULL

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1. Introduction. In a recent paper by Douglas Underwood [**8**] several definitions of "associated prime divisors" were discussed and shown to be unique. In this note we produce a fifth type, which is due to W. Krull, and is found in his classical paper [**2**] and further discussed by B. Banaschewski in [**1**]. Historically this characterization considerably predates the other four definitions.

Throughout this note, R denotes a commutative ring with unity, and all ideals and elements are assumed to be in such a ring. We shall let upper case letters, most frequently the beginning of the alphabet, denote ideals and lower case letters, elements of R. On the whole, our terminology will be that of [**9**]. We do, however, take exception with [**9**] in two instances, viz. $\subset$ will denote set containment which may or may not be proper, and use the symbol $<$ for proper containment. We use the concept "ideal" in the somewhat restrictive sense, in that for us, an ideal is not the entire ring (sometimes in literature this is called a genuine ideal).

2. Preliminary remarks. We first introduce some terminology. Suppose S is a m.c. (nonempty multiplicatively closed) set and A is an ideal. Then the set $\{x \in R |$ there exists $s \in S$, such that $xs \in A\}$ is called the "isolated component of A determined by S" or more simply the S-component of A and is denoted by A_S. If P is a prime and $S = C(P)$ (set complement of P in R), then we let $A_S = A(P)$, i.e., $A(P) = \{x \in R |$ there exists $t \notin P$, such that $xt \in A\}$. (Recall that $A(P) = AR_P \cap R = A^{ec}$.) We shall let $Z(A)$ denote the set of all zero divisors modulo A, i.e., $Z(A) = \{x \in R |$ there exists $y \notin A$, such that $xy \in A\}$. Following the usage of N. McCoy in [**5**], an ideal B is called related to A, if $B \subset Z(A)$, i.e., if each element of B is not prime to A (cf. [**9**, p. 223]). Since the set complement of $Z(A)$ is the collection of all elements in R which are prime to A, it is a multiplicatively closed set, thus by Zorn's Lemma there are ideals contained in $Z(A)$ which contain A and are maximal with respect to this property. These ideals are called maximal associated prime divisors of A, which we shall denote by $MxPD$ (cf. [**7**, p. 19]). The prime ideals minimal in the collection of prime overideals of A are called minimal prime divisors or isolated primes of A. These will be denoted by $MnPD$. If A has a finite primary representation, then the prime ideals which occur as the radicals of the primary ideals in an irredundant representation are called associated primes of A (cf. [**9**, p. 211]).

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We now list the four definitions of "associated prime divisors" as found on p. 72 of [**8**].

(B): P is an associated prime divisor of A (in the Bourbaki sense) if $P = A\colon (x)$ for some $x \in R$.

$(Z\text{–}S)$: P is an associated prime divisor of A (in the Zarski-Samuel sense) if $A\colon (x)$ is a P-primary ideal for some $x \in R$.

(Bw): P is an associated prime divisor of A (in the weak Bourbaki sense), if P is a $MnPD$ of $A\colon (x)$ for some $x \in R$.

(N): P is an associated prime divisor of A (in the Negata sense), if PR_S is a $MxPD$ of AR_S for some multiplicatively closed set S.

Notation. If P is a prime containing an ideal A, then P is called a (B)-prime of A, if P is an associated prime divisor of A in the Bourbaki sense. Similarly for the other conditions.

We now state the theorem of Krull which motivated the definition.

THEOREM [**2**, p. 741]. *If A is an ideal with a finite primary decomposition, then a prime ideal P is one of the associated primes of A if and only if P is a $MxPD$ of $A(P)$.*

W. Krull gave the name *zu A gehörig* to any prime overideal possessing the property: P is an $MxPD$ of $A(P)$, (cf. [**2**, p. 742]) and B. Banaschewski the name *A-extremal* (cf. [**1**, p. 24]). However, in order to be consistent with the terminology due to Underwood we make the following

Definition 1. (K): P is an associated prime divisor of A (in the Krull sense) if P is a $MxPD$ of $A(P)$, thus P is called a (K)-prime of A if and only if P is a $MxPD$ of $A(P)$.

In [**8**] it was shown that if P is a prime containing A, then P a (B)-prime of A implies P is a $(Z\text{–}S)$-prime of A implies P is a (Bw)-prime of A implies P is a (N)-prime of A. We shall show that the (K)-primes of A fit in between the (Bw)-primes and the (N)-primes of A and are in general distinct. But first we record the following which are either easily verified or found in [**1**]:

(1) If U is the set of units in R, then for any ideal A, $Z(A) \cap U = \emptyset$.

(2) Suppose P is a prime containing A. Then $x \in A(P)$ if and only if $x/1 \in AR_P$.

(3) For any prime P, $x \notin P$ implies $x \notin Z(A(P))$, i.e., $(R \backslash P) \cap Z(A(P)) = \emptyset$.

(4) For any ideal A and primes P, P^* containing A, $A \subset A(P)$; $(A(P))(P) = A(P)$; if $P \subset P^*$, then $A(P^*) \subset A(P)$; $(A(P))(P^*) = (A(P^*))(P)$; if $P \subset P^*$, then $(A(P))(P^*) = A(P)$.

We now show that there are several ways to describe the property of being a (K)-prime more precisely.

PROPOSITION 1. *Let A be an ideal and P a prime containing A, then the following statements are equivalent:*

(1) *P is a (K)-prime of A.*
(2) $P \subset Z(A(P))$.
(3) $P = Z(A(P))$.
(4) *PR_P is a $MxPD$ of AR_P.*
(5) *For any multiplicatively closed set T such that $C(P) < T$, then $A_T > A(P)$.*
(6) *For any multiplicatively closed set T such that $A_T \subset A(P)$, then $T \subset C(P)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are clear from the definitions and Remark 3 above. (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (3) are found in [**1**].

(2) $\Rightarrow$ (4): Suppose P is a prime containing A such that $P \subset Z(A(P))$. It is clear that no element in $R_P \backslash PR_P$ belongs to $Z(AR_P)$ (each such element is a unit). So suppose $p/x \in PR_P$ with $p \in P$. Now $p \in Z(A(P))$, so there exists $u \notin A(P)$ such that $pu \in A(P)$. But $u \notin A(P)$ implies $u/1 \notin AR_P$, thus $(p/x)(u/1) = pu/x \in AR_P$, and so $p/x \in Z(AR_P)$, i.e., PR_P is a $MxPD$ of AR_P.

(4) $\Rightarrow$ (2): Suppose PR_P is a $MxPD$ of AR_P and $p \in P$. Then there exists $u/t \in R_P \backslash AR_P$ such that $(p/1)(u/t) = pu/t \in AR_P$. We now show that $u \notin A(P)$ and $pu \in A(P)$. It is easily seen that $u \notin A(P)$, for if so, there exists $v \notin P$ such that $uv \in A$ and then $u/t = uv/tv \in AR_P$. Now since $pu/t \in AR_P$, there exists $a/t' \in AR_P$ such that $a \in A$ and $a/t' = pu/t$, i.e., there exists $s \in P$ such that $ats = put's$, hence $put's \in A$. But since $t', s \notin P$, this implies $t's \notin P$ and so $pu \in A(P)$.

We now prove the result mentioned above, more precisely:

THEOREM 1. *If P is a prime containing A, then*:
(1) *If P is a (Bw)-prime of A, then it is also a (K)-prime of A.*
(2) *If P is a (K)-prime of A, then it is also a (N)-prime of A.*

Proof. (2): Proposition 1, part (4) with the multiplicatively closed set $S = C(P)$.

(1): Now suppose P is a (Bw)-prime of A, then there exists $x \notin A$ such that P is a $MnPD$ of A: (x). But in this case $(A\colon (x))(P)$ is P-primary. Let $p \in P$; then there exists an integer n such that $p^n \in (A\colon (x))(P)$, i.e., there exists $t \notin P$ such that $p^n tx \in A$ or $p^n x \in A(P)$. But since $A\colon (x) \subset P$, this implies that $x \notin A(P)$. Now let k be the least positive integer such that $p^k x \in A(P)$. Clearly $k \neq 0$ and if $k = 1$, then $p \in Z(A(P))$. Suppose that $k > 1$; then $p(p^{k-1}x) \in A(P)$, where $p^{k-1}x \notin A(P)$, i.e., p is a zero divisor modulo $A(P)$. But since p was any element of P, we have that each element of P is a zero divisor modulo $A(P)$, whence P is a (K)-prime of A.

We now exhibit two examples to show that in general the (K)-primes of an ideal are distinct from the (Bw)-primes and (N)-primes. First we find an ideal with an (N)-prime which is not a (K)-prime.

Example 1. Let R be a quasi-local (need not be Noetherian) integral domain containing a field K. By Nagata's theorem [**6**, p. 297] there is a ring T such that:

(i) there exists a maximal ideal M of T such that $T_M \cong R$,
(ii) for each maximal ideal M of T, T_M is isomorphic to R or K,
(iii) the total quotient ring of T is T itself.

Now let M be a maximal prime of T such that $T_M \cong R$. We now show that M is an N-prime of 0 (the zero ideal) but not a (K)-prime of 0. Let S be the m.c. set of units in T. Then by (iii), $T_S \cong T$ and so every element in M is a zero divisor; thus MT_S ($\cong M$) is a $MxPD$ of OT_S ($\cong 0$), hence M is an (N)-prime of 0. It is now seen that M is not a (K)-prime of 0, for by (4) of Proposition 1, M is a (K)-prime of 0 if and only if MT_M is a $MxPD$ of OT_M. But $MT_M \cong R, OT_M \cong 0$, and since R is an integral domain MT_M is not a $MxPD$ of 0.

Example 2. Let us now examine Example 1.3 of [**8**]. Let K be a field of characteristic 0, and R the polynomial ring in infinitely many indeterminates x_i over the field K. Suppose A is the ideal generated by the products x_ix_j for $i \neq j$. It was shown that $M = (x_1, x_2, \ldots)$ is an (N)-prime of A, but not a (Bw)-prime. We shall show that M is a (K)-prime of A, thus showing that the (K) condition on prime overideals does not imply the (Bw) condition. To see this we show that $A(M) = A$ and $Z(A) = M$.

Let $t \notin A$, then $t = a_0 + \sum^{n(1)} a_{1i}x_1{}^i + \ldots + \sum^{n(m)} a_{mi}x_m{}^i + X$ where each monomial term in X has a factor of the form x_ix_j for $i \neq j$. Now since $t \notin A$, either $a_0 \neq 0$ or $a_{ij} \neq 0$ for some i, j. In the first case we have $t = a_0 + [x_i]$, where each monomial term in $[x_i]$ has a factor of x_i for some i. Now if $a_0 = 0$, let p, q be minimal positive integers such that $a_{ij} \neq 0$. In this case $t = a_{pq}x_p{}^q + \sum_{q+1}^{n(p)} a_{pi}x_p{}^i + \ldots + \sum^{n(m)} a_{mi}x_m{}^i + X$. Now for each $u \notin M$, $u = b + [x_i]$, so tu is either $a_0b + [x_i]$, which does not belong to A, or $a_{pq}bx_p{}^q + [x_i]$, which also does not belong to A as $[x_i]$ does not contain any monomial term of the form $cx_p{}^q$. This means $t \notin A(M)$, thus $A = A(M)$, as $A \subset A(M)$ is always true. Now, it is seen that $Z(A) = M$, for suppose $f \in M$, then $f = \sum_1^n q_ix_i$, where $q_i \in R$. But $x_{n+1} \notin A$, and $x_{n+1}f \in A$. This means that $M \subset Z(A(M))$, i.e., M is a (K)-prime of A.

3. Basic properties of (K)-primes. As this note is concerned mostly with (K)-primes of an ideal, we make the following

Definition 2. If A is an ideal, then $k(A)$ is the collection of all (K)-primes of A.

It is well known that if P is a $MnPD$ of A, that $A(P)$ is a P-primary ideal and in this case $P = Z(A(P))$; thus, since every ideal has at least one $MnPD$, $k(A)$ is a nonempty set for any ideal A.

The following remarks are easily verified from the definitions:

1. If P is a prime containing an ideal A and $s \in Z(A(P))$, then $s \in Z(A)$.
2. If P is a prime containing an ideal A, then $x \in A(P)$ if and only if the ideal quotient $A: x$ ($= A: (x)$) is not contained in P.
3. If P is a prime containing A and $x \notin P$, then $A: x \subset A(P)$.

In order to obtain the converse to statement 3, we must restrict ourselves to (K)-primes, i.e., PROPOSITION 2. *If P is a prime containing an ideal A, then $P \in k(A)$ if and only if the following condition holds: $x \notin P$ if and only if $A:x \subset A(P)$.*

Proof. Suppose first the condition holds. Then for each $p \in P$, $A: p \not\subset A(P)$, so there exists $t \notin A(P)$ such that $tp \in A$, whence $p \in Z(A(P))$. So $P \in k(A)$. Conversely, suppose $P \in k(A)$. If $x \notin P$, then x is not related to $A(P)$. Therefore $A(P): x = A(P)$, and since $A \subset A(P)$ we have $A: x \subset A(P): x = A(P)$. Now, if $x \in P \backslash A$, then $x \in Z(A(P))$, so there exists $y \notin A(P)$ such that $xy \in A(P)$. Then there exists $r \notin P$ such that $rxy \in A$. But $yr \notin A(P)$, since $y \notin A(P)$ and $r \notin P$. Thus yr belongs to $A: x$, but not $A(P)$.

In Example 1, M was a maximal ideal which was not a (K)-prime of 0. However, since T is its own total quotient ring, M consists of zero divisors, and thus M is related to the zero ideal. So M is a $MxPD$ of 0 which is not a (K)-prime of 0. Thus, to distinguish between the two possibilities we make the following:

Definition 3. An ideal is said to have the *MBD property* (called an *MDB ideal*), if each of its $MxPD$ are (K)-primes.

Definition 4. An ideal A is said to have the *FB property* (called an *FB ideal*), if it has only a finite number of (K)-primes.

We shall adopt the technique of extending concepts from ideals to rings in the following manner: if T is a property on ideals (e.g., the MDB property), then R has property T if and only if each ideal in R has property T (e.g., R is called a MDB ring if each ideal in R is a MDB ideal). Furthermore, following the lead of Krull, we call a ring in which every ideal has a finite primary decomposition a Lasker ring, more precisely:

Definition 5. An ideal A is called a *Lasker ideal*, if A has a finite primary decomposition, i.e., A is representable as an intersection of finitely many primary ideals.

We now list some of the results found in **[1]** which are relevant to our discussion.

PROPOSITION 3. *For any ideal A, $Z(A) = \bigcup P$ $(P \in k(A))$.*

PROPOSITION 4. *For any ideal A, $A = \bigcap A(P)$ $(P \in k(A))$.*

PROPOSITION 5. *For any ideal A and prime P containing A, $k(A(P)) = \{P^* \in k(A) | P^* \subset P\}$.*

We close this section with a remark explaining why we have talked mainly about the P-components with respect to primes which contain the ideal A.

Remark. If R is a commutative ring with unity, then $A(P) = R$ for a prime P, if and only if $A \not\subset P$.

Proof. If $A \not\subset P$, then $A \cap (R \backslash P) \neq \emptyset$, so $A^e = R_P$, whence $A^{ec} = R$. Conversely if $A(P) = R$, then $1 \in A(P)$, so there exists $x \notin P$ such that $x \cdot 1 \in A$.

4. (*K*)-Primes and intersections. Krull observed that if P is a *MnPD* of A, then the P-component ideal $A(P)$ is P-primary, and in fact, the intersection of all the P-primary ideals which contain A. We now show that the converse is also valid, i.e.,

PROPOSITION 6. *If A is an ideal, then P is a MnPD of A if and only if $A(P)$ is a P-primary ideal.*

Proof. If P is an isolated prime divisor of A, then [**2**, p. 737] shows that $A(P)$ is P-primary. Conversely, if $A(P)$ is P-primary, then P is an isolated prime divisor of A. For if not, then there exists a prime ideal M such that $A \subset M < P$, but then we have $A \subset A(P) \subset A(M) \subset M < P$, and this is impossible unless $M = P$, since $\sqrt{A(P)} = P$.

We now record a useful property that will be used in the sequel.

PROPOSITION 7. *Let $A, B_1, \ldots, B_n$ be ideals in R such that $A = \bigcap^n B_i$. Then for any prime ideal P, $A(P) = \bigcap^n B_i(P)$.*

Proof. An element-wise proof can be constructed. Alternatively the result follows by recalling that $AR_P \cap BR_P = (A \cap B)R_P$ and $A^c \cap B^c = (A \cap B)^c$.

With the aid of the previous observation we can produce two "carry over" results with regard to ideal intersections.

PROPOSITION 8. *If P is a MnPD of both A and B, then P is a (K)-prime of $A \cap B$.*

Proof. If P is an isolated prime of both A and B, then P is an isolated prime of $A \cap B$, hence a (K)-prime of $A \cap B$.

PROPOSITION 9. *If P is a prime such that $P \in k(A)$ and does not contain the ideal B, then $P \in k(A \cap B)$.*

Proof. Since P does not contain B, $B(P) = R$. Thus $(A \cap B)(P) = A(P)$ and so $P \in k(A \cap B)$.

PROPOSITION 10. *For any prime P containing an ideal A, $A(P) = \bigcap A(P^*)$ $(P^* \in k(A)$ and $P^* \subset P)$.*

Proof. The proof follows from Propositions 4 and 5.

We now record another result of Krull.

PROPOSITION 11. *Suppose $A \subset B$ are ideals such that each MxPD of B is contained in a prime P, then $A(P) \subset B$.*

Proof. The result follows from [**2**, p. 733].

Along this same line of thought we have

Proposition 12. *If P is a prime in $k(A)$, and B is an ideal contained between A and $A(P)$, then P is contained in a MxPD of B.*

Proof. Let $x \in P$. Then $x \in Z(A(P))$, so there exists $y \notin A(P)$ such that $xy \in A(P)$. If $xy \in B$, we are through; so assume not. Then there exists $r \notin P$ such that $xyr \in A$. If $yr \notin B$, we are through; so suppose $yr \in B \subset A(P)$. Then there exists $q \notin P$ such that $yrq \in A$; but when r and q do not belong to P, then neither does their product, but this in turn makes $y \in A(P)$, which is a contradiction; so $yr \notin B$ and thus x is an element of $Z(B)$.

It is easy to see that the converses of Propositions 11 and 12 are not true. For let K be a field and $R = K[x, y]$. Suppose $A = (x^2)$, $B = (x^2, xy)$, and $P = (x)$. Then $A(P) = A$ and so $A(P) \subset B$. But y is related to B (since $x \notin B$ and $xy \in B$) and not in P. This shows that the converse of Proposition 11 does not hold, since B must have a *MxPD* which is not contained in P. Now let A, B, and P be as above. Then $B = (x) \cap (x^2, xy, y^2)$. If $P^* = (x, y)$, then $A \subset B \subset P \subset P^*$. But since $A = A(P)$, we do not have B contained in $A(P)$, yet P is contained in a *MxPD* of B.

Proposition 13. *Let A be an ideal with finitely many MxPD's. Then $A = \bigcap A(P)$ (P MxPD of A) is an irredundant intersection.*

Proof. Suppose not, say $P_1, \ldots, P_n$ are *MxPD* of A, $B = \bigcap_2^n A(P_i)$, and $B = A$. Let x be any element of P_1. Then $x \in Z(A)$, so there exists $y \notin A$ such that $xy \in A$. Since $A = B$, this means that $y \notin A(P_j)$ for some $j > 1$, whence $x \in Z(A(P_j))$, which is possible only if $x \in P_j$. So we conclude that $P_1 \subset \bigcup_2^n P_j$, which means $P_1 \subset P_k$ for some $k > 1$. This contradicts the fact that $P_1, \ldots, P_n$ are distinct *MxPD* of A, thus $B \neq A$.

An immediate consequence of this proposition is the fact that when A has a finite number of *MxPD*, but more than one, then necessarily A is a reducible ideal. We can, however, drop the finiteness, as the following observations show:

Lemma. *If A is any ideal with more than one MxPD, then for any MxPD, P, $A \neq A(P)$.*

Proof. Suppose P and P^* are two *MxPD* of an ideal A, then $P \not\subset P^*$, so there exists $x \in P \backslash P^*$. Then $x \in Z(A)$; but $x \notin Z(A(P^*))$, thus $A \neq A(P^*)$.

Proposition 14. *If A is an ideal with more than one MxPD, then A is an intersection (possibly infinite) of proper overideals.*

Proof. The proof is clear since $A = \bigcap_{MxPD} A(P)$ and $A(P) \neq A$ for each *MxPD*, P.

We close this section with a note concerning the *MxPD* of an intersection of component ideals.

Lemma. *Suppose B and C are ideals with only a finite number of MxPD. If $A = B \cap C$, then $P \subset Z(A)$ implies $P \subset Z(B)$ or $P \subset Z(C)$.*

Proof. $Z(B) = \bigcup P^*(P^*, MxPD$ of $B)$ and $Z(C) = \bigcup P^*(P^*, MxPD$ of $C)$. Thus $Z(A) \subset Z(B) \cup Z(C)$; for $x \in Z(A)$ implies there exists $y \notin A$ such that $xy \in A$, and $y \notin A$ means $y \notin B$ or $y \notin C$, so that $x \in Z(B)$ or $x \in Z(C)$. Let $P \subset Z(A)$. Then $P \subset Z(B) \cup Z(C) = \bigcup P^*$ $(P^*, MxPD$ of B or $C)$. This is a finite union of primes so P is contained in at least one of them, i.e., $P \subset Z(B)$ or $P \subset Z(C)$.

PROPOSITION 15. *Let A be an ideal and $P_1, \ldots, P_n$ (K)-primes of A such that each is maximal (with respect to set containment) in the set $\{P_1, \ldots, P_n\}$. Then the $MxPD$ of $B = \bigcap^n A(P_i)$ are precisely these P_i's.*

Proof. We need only consider the case for $n = 2$, as a simple application of mathematical induction can be applied for $n > 2$. Let $P \subset Z(B)$, then $P \subset Z(A(P_1))$ or $P \subset Z(A(P_2))$, which means that $P \subset P_1$ or $P \subset P_2$. Now since $P_i \in k(A)$ and $A \subset B \subset A(P_i)$, we have (Proposition 12) $P_i \subset Z(B)$. Thus the $MxPD$ of B are precisely P_1 and P_2.

5. *MDB* ideals. In this section we turn our attention to displaying properties which insure that an ideal be a MDB ideal. We first note that if A has a unique $MxPD$, say P, then $(R\backslash P) \cap Z(A) = \emptyset$, for if that were not so, there would be another $MxPD$ distinct from P. With this observation we have:

PROPOSITION 16. *If A is an ideal with a unique $MxPD$, then A is a MDB ideal.*

Proof. Let P be the unique $MxPD$ of A and assume the contrary, that is, there exists $p \in P$ such that $p \notin Z(A(P))$. Now $p \in Z(A)$, so that there exists $r \notin A$ such that $rp \in A \subset A(P)$. Since $p \notin Z(A(P))$, this means $r \in A(P)$, so there exists $w \notin P$ such that $wr \in A$. But this is not possible, since $r \notin A$, $wr \in A$, implies $w \in Z(A) \subset P$. Hence $p \in Z(A))$.

We have the following converse:

PROPOSITION 17. *Suppose A is an ideal and P is a (K)-prime of A such that $A = A(P)$, then P is a unique $MxPD$ of A.*

Proof. Since $A = A(P)$ it follows that $Z(A) = Z(A(P))$. But P is a (K)-prime, hence $Z(A(P)) = P$. Thus $Z(A) = P$, so that P is the only $MxPD$ of A.

PROPOSITION 18. *If A is an ideal such that $k(A)$ has only a finite number of maximal elements, then A is an MDB ideal.*

Proof. Let $P_1, \ldots, P_n$ be the maximal elements in $k(A)$ and suppose P is a $MxPD$ of A. Then $P \subset Z(A) = \bigcup^n P_i$, and so $P \subset P_i$ for some i. But since P is a $MxPD$ and $P \subset P_i$, we have $P = P_i$, whence P is a (K)-prime of A.

COROLLARY. *If A is a Lasker ideal, then A is a MDB ideal.*

PROPOSITION 19. *A is a MDB ideal if and only if for each pair of $MxPD$, P, P^*, $x \in P \cap P^*$ implies $x \in Z(A(P)) \cap Z(A(P^*))$.*

Proof. If A is a MDB ideal, the condition is clearly true, so suppose the condition holds and P is a $MxPD$ of A. Let x be any element of P. If there exists a $MxPD$ P^* of A such that $x \in P^*$, then by the condition, x belongs to $Z(A(P))$, so we may assume that no other $MxPD$ of A also contains x. Now $x \in Z(A)$ implies there exists u not belonging to A such that ux belongs to A. Since $x \notin P^*$ for any other $MxPD$, we have $u \in A(P^*)$. Hence $u \notin A(P)$, since $A = \bigcap A(P^*)$ (P^*, $MxPD$ of A). This then means that $x \in Z(A(P))$. But this is true for every $x \in P$, so that $P = Z(A(P))$.

PROPOSITION 20. *If A is an ideal with finitely many $MxPD$ such that for each $MxPD$ P, $A(P)$ also has finitely many $MxPD$, then A is a MDB ideal.*

Proof. $A = \bigcap_{MxPD} A(P^*)$, which is a finite intersection. Furthermore, for each $MxPD$ P^* of A, the $MxPD$ of $A(P^*)$ are contained in P^*. Now using the previous lemma, for each $MxPD$ P of A, $P \subset Z(A(P^*))$ for some P^*. But since P and P^* are $MxPD$ of A and $P \subset P^*$ we have $P = P^*$. This means that $P = Z(A(P))$.

6. Multiplicatively closed sets and component ideals.

We shall now discuss some results pertaining to general m.c. (multiplicatively closed) sets and their associated components of an ideal. The following are immediate consequences of Propositions 1 and 2 of [**1**].

PROPOSITION 21. *If S is a m.c. set not intersecting an ideal A and P is any prime containing A which is disjoint from S, then $A_S(P) = A(P)$.*

PROPOSITION 22. *If S is a m.c. set not intersecting A, then $A_S = \bigcap A(P^*)$, where the P^* are the $MxPD$ of $A(S)$.*

Recall that if S_1 and S_2 are m.c. sets, then so is their intersection. Now suppose A is an ideal and $x \in A_{S_1 \cap S_2}$. Then by definition of the component ideal, there is $t \in S_1 \cap S_2$ such that $tx \in A$. But t belongs to S_1 and S_2, so x belongs to A_{S_1} and A_{S_2}, thus:

PROPOSITION 23. *If S_1 and S_2 are m.c. sets not intersecting and ideal A, then $A_{S_1 \cap S_2} \subset A_{S_1} \cap A_{S_2}$.*

If we consider the special case when the m.c. sets are complements of primes containing A, then we have the converse containment relation:

PROPOSITION 24. *If P_1 and P_2 are primes containing A, then letting $S = (P_1 \cup P_2)^c$, we have $A_S = A(P_1) \cap A(P_2)$. So if we call A_S, $A(P_1 \cup P_2)$, we have $A(P_1 \cup P_2) = A(P_1) \cap A(P_2)$.*

Proof. Since $S = P_1{}^c \cap P_2{}^c$, we have by the previous proposition, $A(P_1 \cup P_2) \subset A(P_1) \cap A(P_2)$. Now let x belong to $A(P_1) \cap A(P_2)$. Then for $i = 1, 2$ there exists t_i not belonging to P_i such that xt_i belongs to A. If $t_1 \notin P_2$ or

$t_2 \notin P_1$ we are done, so suppose not. Then it is seen that $t_1 + t_2$ does not belong to P_1 nor P_2, thus x belongs to $A(P_1 \cup P_2)$ since $x(t_1 + t_2) \in A$.

In fact, if S is a m.c. set, such that its set complement is a finite union of prime ideals ("finitely saturated"), we have

PROPOSITION 25. *Let* $P_1, \ldots, P_n$ *be a collection of prime ideals containing* A *and* $S = \bigcap^n P_1^c$. *Then* $A(S) = \bigcap^n A(P_i)$.

Proof. By Proposition 5, $A_S = \bigcap A(P)$, where the intersection is taken over the $MxPD$ of A_S. Now P is a $MxPD$ of A_S, $P \cap S = \emptyset$, so $P \subset \bigcup^n P_i$. But this means that $P \subset P_i$ for some i, which in turn means that $A(P_i) \subset A(P)$, so that we have $\bigcap^n A(P_i) \subset A(P_i) \subset A(P)$. But this is true for each $MxPD$, so that $\bigcap^n A(P_i) \subset \bigcap_{MxPD} A(P)$. Furthermore $A_S \subset A(P_i)$ for each i, so we have the statement $A_S \subset (\bigcap^n A(P_i)) \subset (\bigcap_{MxPD} A(P)) = A_S$.

We shall close our remarks with a result resembling the Cohen theorem (every ideal is finitely generated if and only if every prime ideal is finitely generated) in that in the special case of rings in which every ideal has only a finite number of $MxPD$, a property on the set complement of prime ideals guarantees it for all m.c. sets. To this end we make the following

Definition 6 (Krull). An ideal A, is called a q-ideal, if for each m.c. set S not intersecting A, there is an element x such that $A_S = A\colon x$.

Definition 7. An ideal A is called a strong k-ideal, if for each prime ideal P, there is an element s such that $A(P) = A\colon s$.

PROPOSITION 26. *Suppose* A *is a MDB ideal with finitely many* $MxPD$. *Then* A *is a* q*-ideal if and only if* A *is a strong* k*-ideal.*

Proof. Clearly A a q-ideal implies that A a strong k-ideal. So suppose S is a m.c. set not intersecting A. Then $A_S = \bigcap A(P_i)$ $(MxPD)$. But A is a strong k-ideal, so that for each i there exists x_i such that $A(P_i) = A\colon x_i$. Since R is a MDB ideal, $x_i \notin P_i$ (cf. Proposition 2). Now $\bigcap A(P_i) = \bigcap A\colon x_i = A\colon (x_1, \ldots, x_n)$, but $x_i \notin P_i$ implies $(x_1, \ldots, x_n)$ is not contained in $\bigcup P_i$. So let r belong to $(x_1, \ldots, x_n)$ but not to $\bigcup P_i$. It is now seen that $A_S = A\colon r$, since $A_S \subset A\colon r$ and $t \in A\colon r$ implies $t \in A(P_i)$ for each i, and so $t \in \bigcap A(P_i) = A_S$.

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ON THE β-CONSTRUCTION IN K-THEORY

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Introduction. The β-construction assigns to each complex representation φ of the compact Lie group G a unique element $\beta(\varphi)$ in $\tilde{K}^1(G)$. For the details of this construction the reader is referred to [**1**] or [**5**]. The purpose of the present paper is to determine some of the properties of the element $\beta(\varphi)$ in terms of the invariants of the representation φ. More precisely, we consider the following question. Let G be a simple, simply-connected compact Lie group and let $f: S^3 \rightarrow G$ be a Lie group homomorphism. Then $\tilde{K}^1(S^3) \simeq \mathbf{Z}$ with generator $x = \beta(\varphi_1)$, φ_1 the fundamental representation of S^3, so that if φ is a representation of $G, f^*(\varphi) = n(\varphi)x$, where $n(\varphi)$ is an integer depending on φ and f. The problem is to determine $n(\varphi)$.

Since G is simple and simply-connected we may assume that ch_2, the component of the Chern character in dimension 4 takes its values in $H^4(SG, \mathbf{Z}) \cong \mathbf{Z}$. Let u be a generator of $H^4(SG, \mathbf{Z})$ so that $\mathrm{ch}_2(\beta(\varphi)) = m(\varphi)u$, $m(\varphi)$ an integer depending on φ. It turns out that the integers $n(\varphi)$ and $m(\varphi)$ are closely related to an invariant of the representation φ studied by Dynkin in [**4**] and called by him the index of φ, $l(\varphi)$. In section 2 it is shown that $n(\varphi)$ is a fixed integral multiple, depending on f, of $l(\varphi)$, and in section 3 $m(\varphi)$ is shown to be $\pm l(\varphi)$.

The above may be interpreted as giving information about the question of homotopy of group representations. If we say that two representations φ_1, φ_2 are stably homotopic ($\varphi_1 \sim \varphi_2$) if $i_1 \circ \varphi_1$ and $i_2 \circ \varphi_2$ are homotopic as maps $G \rightarrow \mathrm{Aut}(\mathbf{C}^N)$, N large, i_1, i_2 suitable inclusions, then $\varphi_1 \sim \varphi_2$ if and only if $\beta(\varphi_1) = \beta(\varphi_2)$. It is a consequence of a result of Hodgkin [**5**] that given an arbitrary φ, $\varphi \sim n_1\lambda_1 + \ldots + n_k\lambda_k$, $\lambda_1, \ldots, \lambda_k$ the fundamental representations of G and $n_1, \ldots, n_k$ suitable integers. In particular inequivalent representations may be stably homotopic. In these terms the above says that a necessary condition that two representations be stably homotopic is that their indices be equal. Since a theorem of Dynkin (1.4 below) gives the index of an irreducible φ as a simple expression involving the dimensions of φ and G and the highest weight of φ, this criterion is a useful one for studying the question of stable homotopy.

The contents of this paper are arranged in the following way. Section 1 develops the properties of the index in the context of representations and homomorphisms of complex simple Lie algebras. This material is essentially all contained in [**4**]. In section 2 we establish the connection between the index and the β-construction. This section also discusses the question of stable homotopy

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of representations. The final section gives the relationship between the index and the Chern character.

1. The Index. Let $\mathfrak{g}$ be a complex simple Lie algebra. Then $\mathfrak{g}$ admits an invariant inner product (e.g. the Killing form) and further any two such differ by a scalar factor. Take $\mathfrak{h} \subset \mathfrak{g}$ to be a fixed Cartan subalgebra and let $R \subset \mathfrak{h}^*$ be the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. An inner product on $\mathfrak{h}$ determines one on $\mathfrak{h}^*$ by duality; it follows that we obtain a unique invariant inner product, which will be denoted as $\mathfrak{g}(\cdot,\cdot)$ by imposing the normalization condition $\mathfrak{g}(\mu, \mu) = 2$ where $\mu \in R$ is the maximal root with respect to some ordering of R. Given the form $\mathfrak{g}(\cdot,\cdot)$ and a root $\alpha \in R$ we denote as h_α the element dual to α, that is, the unique element $h_\alpha \in \mathfrak{h}$ such that $\mathfrak{g}(h_\alpha, X) = \alpha(X)$ for all $X \in \mathfrak{h}$.

Now let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be simple Lie algebras and let $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism of Lie algebras. Then the formula $(x, y) = \mathfrak{g}_2(f(x), f(y))$ defines an invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}_1$. By the above remarks, $(\cdot,\cdot) = j(f)\mathfrak{g}_1(\cdot,\cdot)$ for a fixed scalar $j(f)$. Following Dynkin [**4**] we define the index of f to be the scalar $j(f)$. It is clear that $j(f) = \frac{1}{2}\mathfrak{g}_2(f(h_\mu), f(h_\mu))$, μ the maximal root of g_1 and that if $g\colon \mathfrak{g}_2 \to \mathfrak{g}_3$ is a second homomorphism then $j(g \circ f) = j(f)j(g)$. More difficult is the fact ([**4**], Theorem 2.2) that $j(f)$ is always integral. Although not central to the present work, 3.3 below leads easily to an independent proof of this fact. For later reference we also record the following obvious proposition.

PROPOSITION 1.1. *Let μ be the maximal root of g with respect to some ordering of R and let X_μ, $X_{-\mu}$ be root vectors for μ and $-\mu$ respectively. Then the subalgebra spanned by $\{h_\mu, X_\mu, X_{-\mu}\}$ is isomorphic to A_1 and if $f\colon A_1 \to \mathfrak{g}$ denotes the corresponding inclusion, then $j(f) = 1$.*

Now let $\varphi\colon \mathfrak{g} \to \mathrm{End}(\mathbf{C}^n)$ be a representation of $\mathfrak{g}$. The trace form $\mathrm{Tr}_\varphi(\cdot,\cdot)$ defined by $\mathrm{Tr}_\varphi(x, y) = \mathrm{Tr}(\varphi(x) \circ \varphi(y))$ is again an invariant inner product on $\mathfrak{g}$ so that once again $\mathrm{Tr}_\varphi(\cdot,\cdot) = l(\varphi)\mathfrak{g}(\cdot,\cdot)$ where $l(\varphi)$ is a scalar called by Dynkin the index of φ. As immediate consequences of the definition of $l(\varphi)$ we have the following propositions.

PROPOSITION 1.2. *If $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism and $\varphi\colon \mathfrak{g}_2 \to \mathrm{End}(\mathbf{C}^n)$ a representation, then $l(\varphi \circ f) = j(f)l(\varphi)$.*

PROPOSITION 1.3. *If $\varphi\colon \mathfrak{g} \to \mathrm{End}(\mathbf{C}^n)$, $\psi\colon \mathfrak{g} \to \mathrm{End}(\mathbf{C}^m)$ are two representations, then $l(\varphi + \psi) = l(\varphi) + l(\psi)$ and $l(\varphi \otimes \psi) = ml(\varphi) + nl(\psi)$ (for the latter property one requires the fact that $\mathfrak{g}$ is simple).*

The following theorem [**4**, Theorem 2.5] gives an elegant expression for $l(\varphi)$ in terms of the invariants of φ. For completeness we include a proof.

THEOREM 1.4. *Let φ be an irreducible representation of the simple Lie algebra $\mathfrak{g}$. Then $l(\varphi) = (\dim \varphi/\dim \mathfrak{g})\mathfrak{g}(\lambda, \lambda + 2\delta)$, where λ is the highest weight of φ and δ is $\frac{1}{2}$ the sum of the positive roots.*

Proof. Let l be the dimension of $\mathfrak{h}$ and choose a basis $\{H_i\}_{i=1}^{l}$ of $\mathfrak{h}$ orthonormal with respect to $\mathfrak{g}(\cdot,\cdot)$. We now choose a root vector X_α corresponding to each $\alpha \in R$, subject to the condition (to be used later) that $[X_\alpha, X_{-\alpha}] = h_\alpha \mathfrak{g}(X_\alpha, X_{-\alpha})$. Then $\{H_i\}_{i=1}^{l} \cup \{X_\alpha\}_{\alpha\in R}$ is a basis for $\mathfrak{g}$. Recalling that $g(X_\alpha, X_\beta) = 0$ unless $\alpha + \beta = 0$ and that $\mathfrak{g}(H_i, X_\alpha) = 0$ all i, α, it follows that

$$\left\{\frac{H_i}{l(\varphi)}\right\}_{i=1}^{l} \cup \left\{\frac{X_{-\alpha}}{l(\varphi)\mathfrak{g}(X_\alpha, X_{-\alpha})}\right\}_{\alpha\in R}$$

is dual to the above basis with respect to $\mathrm{Tr}_\varphi(\cdot,\cdot)$. ($l(\varphi) \neq 0$ since $\mathfrak{g}$ simple, $\varphi \neq 0 \Rightarrow \mathrm{Tr}_\varphi(\cdot,\cdot)$ non-degenerate.) Since $\mathfrak{g}$ is simple and φ is irreducible, it is a standard fact (cf. [**6**, III.4]) that the Casimir operator

$$\Phi = \frac{1}{l(\varphi)}\left[\sum_{i=1}^{l} \varphi(H_i)^2 + \sum_{\alpha\in R} \frac{\varphi(X_\alpha)\varphi(X_{-\alpha})}{\mathfrak{g}(X_\alpha, X_{-\alpha})}\right]$$

is a scalar multiple of the identity and that $\mathrm{Tr}\Phi = \dim \mathfrak{g}$. On the other hand if v is a vector of weight λ the relations $\varphi(H_i)v = \lambda(H_i)v$ and

$$\frac{\varphi(X_\alpha)\varphi(X_{-\alpha})}{\mathfrak{g}(X_\alpha, X_{-\alpha})}v = \begin{cases} 0, & \alpha < 0 \\ \dfrac{\lambda([X_\alpha, X_{-\alpha}])}{\mathfrak{g}(X_\alpha, X_{-\alpha})}v = \mathfrak{g}(\lambda, \alpha)v, & \alpha > 0 \end{cases}$$

show that $\Phi v = (1/l(\varphi))\mathfrak{g}(\lambda, \lambda + 2\delta)v$ and thus that

$$\mathrm{Tr}\Phi = (\mathfrak{g}(\lambda, \lambda + 2\delta)/l(\varphi))\dim \varphi.$$

Theorem 1.4 follows.

Dynkin [**4**, table 5, p. 135] has used 1.4 to compute $l(\varphi)$ when φ is a fundamental representation of any of the simple algebras. In particular if $\mathfrak{g} = A_n$ and φ is the representation of A_n as $\mathrm{sl}(n + 1, \mathbf{C})$, then

(1.5) $$l(\varphi) = 1 \quad \text{and} \quad l(\Lambda^k\varphi) = \binom{n-1}{k-1}.$$

COROLLARY 1.6. *Let φ be any representation of the simple Lie algebra $\mathfrak{g}$. Then*

$$l(\Lambda^k\varphi) = \binom{\dim \varphi - 2}{k - 1} l(\varphi).$$

The next corollary is a consequence of 2.2 below together with the properties of the Adams operations in $\tilde{K}^0(S^4)$. Let φ be a representation of $\mathfrak{g}$ and let $\psi^k(\varphi)$ be the virtual representation defined in a fashion analogous to the definition of the Adams operations in K-theory, i.e., let $P_k(\sigma_1, \ldots, \sigma_k)$ be the Newton polynomial expressing $x_1^k + \ldots + x_k^k$ in terms of the elementary symmetric functions in $x_1, \ldots, x_k$ and define $\psi^k(\varphi) = P_k(\varphi, \Lambda^2\varphi, \ldots, \Lambda^k\varphi)$.

COROLLARY 1.7. $l(\psi^k(\varphi)) = k^2 l(\varphi)$.

2. The relation of the index to K-theory. Let G be a simple and simply-connected compact Lie group and let $f\colon S^3 \to G$, $\varphi\colon G \to \mathrm{Aut}(\mathbf{C}^n)$, be a homomorphism and representation respectively. We denote as $\tilde{f}$, $\tilde{\varphi}$ the associated homomorphism and representation of (complex) Lie algebras. The β-construction associates to each such φ an element $\beta(\varphi) \in \tilde{K}^1(G)$. Recall [**5**, §4] that the β-construction has the properties

(2.1) $\beta(\varphi + \psi) = \beta(\varphi) + \beta(\psi)$ and $\beta(\varphi \otimes \psi) = \dim \psi\beta(\varphi) + \dim \varphi\beta(\psi)$.

Recall also that $\tilde{K}^1(S^3) \simeq \mathbf{Z}$ with generator $x = \beta(\varphi_1)$ where φ_1 is the representation of S^3 corresponding to the representation of A_1 as $\mathrm{sl}(2, \mathbf{C})$. The main result of this section is the following theorem.

THEOREM 2.2. $f^*\beta(\varphi) = l(\tilde{\varphi} \circ \tilde{f})x = j(\tilde{f})l(\tilde{\varphi})x.$

Proof. $\varphi \circ f$ is a representation of S^3 so that $f^*\beta(\varphi) = \beta(\varphi \circ f)$; therefore by 1.2 it is enough to show that if ψ is any representation of S^3, $\tilde{\psi}$ the associated representation of A_1, then $\beta(\psi) = l(\tilde{\psi})x$. Since $l(\tilde{\varphi}_1) = 1$ (1.5) and since any such ψ may be written as a polynomial in φ_1, 2.2 now follows from 1.3 and 2.1.

Let $\varphi\colon G \to \mathrm{Aut}(\mathbf{C}^n)$, $\psi\colon G \to \mathrm{Aut}(\mathbf{C}^m)$ be two representations and define φ to be stably homotopic to ψ, written $\varphi \sim \psi$, if $i_1 \circ \varphi$ and $i_2 \circ \psi$ are homotopic as maps, where $i_1\colon \mathrm{Aut}(\mathbf{C}^n) \to \mathrm{Aut}(\mathbf{C}^N)$ and $i_2\colon \mathrm{Aut}(\mathbf{C}^m) \to \mathrm{Aut}(\mathbf{C}^N)$ are induced by inclusion and N is suitably large. From the characterization of the β-construction given in [**5**, §4] it is clear that $\varphi \sim \psi$ if and only if $\beta(\varphi) = \beta(\psi)$. The following is thus a corollary of Theorems 2.2 and 1.4.

COROLLARY 2.3. *Let φ, ψ be two irreducible representations of G with highest weights λ, λ' respectively. Then a necessary condition that φ be stably homotopic to ψ is that* $\dim \varphi(\lambda, \lambda + 2\delta) = \dim \psi(\lambda', \lambda' + 2\delta)$ *(here $(\cdot,\cdot)$ denotes any invariant inner product on $\mathfrak{g}$).*

Remark. The above condition is clearly not sufficient. Since every representation of S^3 is self-conjugate the above cannot distinguish a representation from its dual. However if φ is the representation of $\mathrm{SU}(n + 1)$ defined by $\varphi\colon \mathrm{SU}(n + 1) \subset \mathrm{Aut}(\mathbf{C}^n)$ then $\Lambda^k\varphi$ and $\Lambda^{n-k+1}\varphi$ are dual but not (unless $k = n - k + 1$) stably homotopic. I do not know an example of two distinct, non-conjugate, irreducible representations φ, ψ with $l(\varphi) = l(\psi)$ nor any example of two distinct irreducible representations φ, ψ with $\varphi \sim \psi$.

Corollary 2.3 together with the remarks made above does give the following result concerning the geometry of the Coxeter-Stiefel diagram.

COROLLARY 2.4. *Let φ be an irreducible representation of the simple Lie algebra $\mathfrak{g}$ and let λ be the highest, λ' the lowest, weight of φ. Then*

$$(\lambda, \lambda + 2\delta) = (-\lambda', -\lambda' + 2\delta).$$

Proof. $-\lambda'$ is the highest weight of the dual (contragredient) representation.

3. The relationship between the Chern character and the index.

To each $x \in \tilde{K}^1(G)$ the Chern character associates an element $\mathrm{ch}_2 x \in H^4(SG, \mathbf{Q})$.

We consider $H^4(SG, \mathbf{Z})$ as a subgroup of $H_4(SG, \mathbf{Q})$ under the coefficient homomorphism $\mathbf{Z} \subset \mathbf{Q}$ and consider $H^4(SG, \mathbf{Z})$ as those $u \in H^4(SG, \mathbf{Q})$ for which $\langle u, v\rangle \in \mathbf{Z}$ whenever $v \in H_4(SG, \mathbf{Z})$. If G is simple and simply-connected $H_4(SG, \mathbf{Z}) \simeq \mathbf{Z}$ has all its elements spherical. Since the Chern character takes integral values on spherical cycles (cf. [**2**]) we may, by the above, consider ch_2 as a map $\mathrm{ch}_2\colon \tilde{K}^1(G) \to H^4(SG, \mathbf{Z})$. We will need the following proposition.

PROPOSITION 3.1. *There is a homomorphism* $f\colon S^3 \to G$ *such that* $[f]$ *generates* $\pi_3(G) \simeq \mathbf{Z}$ *and such that* $j(\tilde{f}) = 1$. (*$[f]$ denotes the homotopy class of f.*)

Proof. This is a consequence of Proposition 10.2 Chapter III of [**3**]; in particular, the statement about $j(\tilde{f})$ follows from the proof of part B of this proposition together with 1.1 above.

Now let x be the generator of $\tilde{K}^1(S^3)$ described in §2 and let ι be the generator of $H_4(S^4, \mathbf{Z})$ with $\langle \mathrm{ch} x, \iota\rangle = 1$. Let $v = f_*(\iota)$ generate $H_4(SG, \mathbf{Z})$ (f the homomorphism of 3.1) and choose as generator of $H^4(SG, \mathbf{Z})$ the element u such that $\langle u, v\rangle = 1$.

THEOREM 3.2. (a) *Let* φ *be a representation of* G. *Then*

$$\mathrm{ch}_2\beta(\varphi) = l(\varphi)u.$$

(b) *The image of* $\mathrm{ch}_2\colon \tilde{K}^1(G) \to H^4(SG, \mathbf{Z})$ *is generated by* ku *where*

$$k = \begin{cases} 1, & G = \mathrm{SU}(n) \text{ or } \mathrm{Sp}(n) \\ 2, & G = \mathrm{Spin}\, n,\ n = 5,\ n \geqq 7, \text{ or } G_2 \\ 6, & G = F_4 \text{ or } E_6 \\ 12, & G = E_7 \\ 20, & G = E_8. \end{cases}$$

Proof. $\mathrm{ch}_2\beta(\varphi) = m(\varphi)u$ where $m(\varphi) = \langle \mathrm{ch}_2\beta(\varphi), v\rangle$. But $\langle \mathrm{ch}_2\beta(\varphi), v\rangle = \langle \mathrm{ch}_2\beta(\varphi \circ f), \iota\rangle = l(\varphi)\langle \mathrm{ch}_2 x, \iota\rangle = l(\varphi)$.

Part (b) follows from the computations of Dynkin ([**4**], Table 5).

Similar computations now yield the following.

COROLLARY 3.3. *Let* $g\colon S^3 \to G$ *be any homomorphism. Then in* $\pi_3(G)$, $[g] = j(\tilde{g})[f]$, f *as above.*

Remark. As noted earlier this corollary leads to a proof of the fact that $j(f)$ is always integral. This proof makes no use of 1.4 but does require several deep topological results, in particular 3.1 and the integrality of the Chern character.

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CERTAIN SUBSETS OF PRODUCTS OF METACOMPACT SPACES AND SUBPARACOMPACT SPACES ARE REALCOMPACT

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We will say that a space X has property $(*)$ if and only if each discrete subset of X is realcompact; i.e., the cardinality of each discrete subset of X is nonmeasurable. In [**8**], Shirota shows that a completely regular T_1-space X is realcompact if and only if X has property $(*)$ and X is complete with respect to some uniformity. In [**7**], Moran, using measure theoretic techniques, shows that any normal metacompact T_1-space with property $(*)$ is realcompact.

Let $\mathscr{M}$ denote the class of T_3-spaces which are either metacompact or subparacompact. It is the purpose of this paper to establish the following result:

THEOREM. *The normal space X is realcompact if and only if X has property $(*)$ and X can be embedded as a closed subspace in the product of a collection of members of $\mathscr{M}$.*

Recall that a space X is *realcompact* if it can be embedded as a closed subspace in the product of a collection of copies of the reals. For basic theorems and notation concerning realcompact spaces and ultrafilters, the reader is referred to [**3**].

The space X is said to be *metacompact* if whenever $\mathscr{U}$ is an open cover of X, there is a point finite open refinement of $\mathscr{U}$

A space is said to be *subparacompact* if whenever $\mathscr{U}$ is an open cover of X, there is a refinement $\mathscr{H} = \bigcup_{i=1}^{\infty} \mathscr{H}_i$ of $\mathscr{U}$ that covers X such that, for each i, $\mathscr{H}_i$ is a discrete collection of closed sets. The name "subparacompact" is due to Burke [**1**]. In [**5**], McAuly shows that every semi-metric space is subparacompact and in [**2**] Creede shows that every semi-stratifiable space is subparacompact.

It should be pointed out that the normality condition of our Theorem cannot be removed since the space Ψ given in Exercise 51 of [**3**] is a completely regular Moore space that is not realcompact.

For convenience, if $\mathscr{E}$ is a class of spaces and X is a space, then $C(X, \mathscr{E})$ will denote the class of continuous functions on X with range in $\mathscr{E}$. If f is a function, then Range(f) will denote the range of f.

LEMMA 1. *Suppose that X is a T_1-space and $\mathscr{E}$ is a class of T_3-spaces such that the topology on X is the weak topology induced by $C(X, \mathscr{E})$. Then X can be*

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embedded as a closed subspace in the product of a collection of members of $\mathscr{E}$ *if if and only if it is true that, if* $\mathscr{F}$ *is a free ultrafilter of closed subsets of* X, *then there is a member* f *of* $C(X, \mathscr{E})$ *and an open cover* $\mathscr{U}$ *of* Range(f) *such that* $\{f^{-1}(U) : U \in \mathscr{U}\}$ *refines* $\{(X - F) : F \in \mathscr{F}\}$.

Proof. Suppose that $\mathscr{E}'$ is a collection of members of $\mathscr{E}$ and T is a homeomorphism taking X onto a closed subspace of $\Pi\mathscr{E}'$. For each $E \in \mathscr{E}'$, P_E will denote the projection of $\Pi\mathscr{E}'$ onto E; thus, for each $E \in \mathscr{E}'$, $P_E \cdot T$ is in $C(X, \mathscr{E})$. Let $\mathscr{F}$ be a free ultrafilter of closed subsets of X. For each E in $\mathscr{E}'$, let $\mathscr{F}_E$ denote the set, $\{H : H$ is a closed subset of E and $(P_E \cdot T)^{-1}(H) \in \mathscr{F}\}$.

Claim 1. If $E \in \mathscr{E}'$, then there is at most one point in $\bigcap \mathscr{F}_E$: since E is regular, if there were two points p and q in $\bigcap\mathscr{F}_E$, then there would be open sets U and V in E containing p and q respectively such that $U^- \cap V^- = \emptyset$; thus, $(P_E \cdot T)^{-1}(U^-)$ and $(P_E \cdot T)^{-1}(V^-)$ would be disjoint members of $\mathscr{F}$, which would be impossible.

Claim 2. Suppose that $E \in \mathscr{E}'$ and x is a point of E such that if U is an open set in E that contains x, then U^- is in $\mathscr{F}_E$. Then x is in each member of $\mathscr{F}_E$. To see that this is true, suppose the contrary; i.e., suppose that there is a member H of $\mathscr{F}_E$ that does not contain x. By the regularity of E, there is an open set U containing x such that $U^- \cap H = \emptyset$, but this is impossible since $\mathscr{F}_E$ is centered.

Claim 3. There is a member E of $\mathscr{E}'$ such that $\bigcap\mathscr{F}_E = \emptyset$: suppose otherwise. Then there is a point x of $\Pi\,\mathscr{E}'$ such that, for each E, $P_E(x) = \bigcap\mathscr{F}_E$. We will show that x is a limit point of $T(X)$; and so, $T^{-1}(x)$ will be a member of $\mathscr{F}$, which will be a contradiction. To this end, let $\mathscr{E}''$ be a finite subset of $\mathscr{E}'$ and for each E in $\mathscr{E}''$, let U_E be an open set in E containing $P_E(x)$. It will suffice to show that there is a point y of X such that $P_E(T(y))$ is in U_E for each E in $\mathscr{E}''$. For each E in $\mathscr{E}''$, let V_E be an open set in E containing $P_E(x)$ such that $V_E^- \subset U_E$. Then

$$\bigcap_{E \in \mathscr{E}''} (P_E \cdot T)^{-1}(V_E^-) \neq 0;$$

and so, let y be a point of this intersection. It must be the case that $P_E(T(y))$ is in U_E for each E in $\mathscr{E}''$.

Let E be a member of $\mathscr{E}'$ such that $\bigcap\mathscr{F}_E = \emptyset$. According to Claim 2, there is an open cover $\mathscr{U}$ of E such that $\{(P_E \cdot T)^{-1}(U^-) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$, which completes this part of the proof.

Suppose now that it is true that: if $\mathscr{F}$ is a free ultrafilter of closed sets in X, then there are a member f of $C(X, \mathscr{E})$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let $\mathscr{E}'$ be a collection of members of $\mathscr{E}$ such that : (*i*) $C(X, \mathscr{E}')$ determines the topology on X and (ii) if $\mathscr{F}$ is a free ultrafilter of closed sets in X, then there are a member f of $C(X, \mathscr{E}')$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let T be the natural embedding of X into

$$\prod_{f \in C(X, \mathscr{E}')} \text{Range}(f)$$

given by $P_f(T(x)) = f(x)$ (see the Embedding Lemma, [**4**, p. 116]). Let y be a limit point of $T(X)$ in

$$\{\prod_{f\in C(X,E')} \mathrm{Range}(f)\} - T(X).$$

Let $\mathscr{F}' = \{f^{-1}(V^{-}) : V$ is an open set in Range (f) containing $P_f(y), f \in C(X, \mathscr{E}')\}$. Then $\mathscr{F}'$ is a centered collection in X. Since y is not in $T(X)$, it must be the case that $\bigcap \mathscr{F}' = \emptyset$. Let $\mathscr{F}$ be an ultrafilter of closed sets in X that contains $\mathscr{F}'$. $\mathscr{F}$ must be free; and so, there is a member f of $C(X, \mathscr{E}')$ and an open cover $\mathscr{U}$ of Range(f) such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$. Let V be an open set in Range(f) containing $P_f(y)$ such that V^{-} is a subset of some member of $\mathscr{U}$. Then $f^{-1}(V^{-})$ is not in $\mathscr{F}$, which is a contradiction from which the lemma follows.

LEMMA 2 [**6**, Theorem 18]. *If $\mathscr{U}$ is an open cover of the space X, then there is a discrete subset H of X such that*

(1) $\{st(x, \mathscr{U}) : x \in H\}$ *covers* X

and

(2) *No member of $\mathscr{U}$ contains two points of H.*

In the following theorem, $\mathscr{M}_1$ will denote the class of regular metacompact T_1-spaces; $\mathscr{M}_2$ will denote the class of regular subparacompact T_1-spaces; and $\mathscr{M}_3 = \mathscr{M}_1 \cup \mathscr{M}_3$.

THEOREM. *The following conditions on a normal T_1-space are equivalent:*

(1) *X is realcompact.*

(2) *X has property $(*)$ and X can be embedded as a closed subspace in the product of a collection of members of $\mathscr{M}_3$.*

(3) *X has property $(*)$ and if $\mathscr{F}$ is a free ultrafilter of closed subsets of X, then there is a point finite open cover of X that refines $\{X - F : F \in \mathscr{F}\}$.*

Proof. (1) $\Rightarrow$ (2): This is obvious since every closed subset of a realcompact space is realcompact and the real line is a member of $\mathscr{M}_3$.

(2) $\Rightarrow$ (1): Let $\mathscr{Z}$ be a free z-ultrafilter in X. We will show that X is realcompact by showing that there is a countable subcollection of $\mathscr{Z}$ with empty intersection.

Let $\mathscr{F}$ be an ultrafilter of closed subsets of X that contains $\mathscr{Z}$. Since X can be embedded as a closed subset of the product of a collection of members of $\mathscr{M}_3$, by Lemma 1, there is a member M of $\mathscr{M}_3$, a map f taking X onto M, and an open cover $\mathscr{U}$ of M such that $\{f^{-1}(U) : U \in \mathscr{U}\}$ refines $\{X - F : F \in \mathscr{F}\}$.

Case 1. M is in $\mathscr{M}_1$: Let $\mathscr{U}'$ be a point finite open refinement of $\mathscr{U}$ that covers M. Applying Lemma 2, there is a discrete subset H of X such that

(i) $\{st(x, f^{-1}(\mathscr{U}')) : x \in H\}$ covers X and

(ii) no member of $f^{-1}(\mathscr{U}')$ contains two points of H.

Let $\mathscr{W}$ be the subcollection of $f^{-1}(\mathscr{U}')$ consisting of all W which contain a point of H. Now, since $\mathscr{W}$ is both point finite and infinite, it follows that the cardinality of $\mathscr{W}$ is the same as the cardinality of H; and so, there is a one-to-one function φ from H to $\mathscr{W}$. For each F in $\mathscr{F}$, let $H(F) = \{x \in H : \varphi(x) \cap F = \emptyset\}$. Clearly, $\{H(F) : F \in \mathscr{F}\}$ has the finite intersection property; and so, we may let $\mathscr{K}$ denote an ultrafilter of subsets of H that contains $\{H(F) : F \in \mathscr{F}\}$. Since, for each $x \in H$ there is an F_x in $\mathscr{F}$ such that $\varphi(x) \cap F_x = \emptyset$, $\mathscr{K}$ must be a free ultrafilter in H. Thus, since H is realcompact, there is a countable subcollection $\{K_1, K_2, \ldots\}$ of $\mathscr{K}$ with $\bigcap K_i = \emptyset$.

Claim 1. If $K \in \mathscr{K}$, then there is a member F of $\mathscr{F}$ such that $F \subset \bigcup_{x\in K}(\varphi(x))$: To see that this is so, suppose otherwise; i.e., suppose that there is a K in $\mathscr{K}$ such that $X - \bigcup_{x\in K}(\varphi(x))$ is in $\mathscr{F}$. Then $H(X - \bigcup_{x\in K} (\varphi(x)))$ is a member of $\mathscr{K}$ that fails to intersect K which is a contradiction and hence Claim 1 follows.

Claim 2. $\bigcap_{i=1}^{\infty}((\bigcup_{x\in K_i}(\varphi(x))) = \emptyset$: To see that this is true, let $y \in X$ and observe that there is a finite subset S of H such that $\varphi(x)$ contains y if and only if $x \in S$. Thus, since $\bigcap K_i = \emptyset$, there is an integer n such that $K_n \cap S = \emptyset$. For that n, it must be the case that $\bigcup_{x\in K_n}(\varphi(x))$ does not contain y.

Now, By Claim 1, for each integer n there is a member F_n of $\mathscr{F}$ such that $F_n \subset \bigcup_{x\in K_n}(\varphi(x))$. Since X is normal, for each n there is a zero-set, Z_n, in X such that $F_n \subset Z_n \subset \bigcup_{x\in K_n}(\varphi(x))$. $\{Z_1, Z_2, \ldots\}$ is a subcollection of $\mathscr{Z}$ with $\bigcap Z_i = \emptyset$.

Case 2. M is in M_2: Let $\mathscr{H} = \bigcup_{i=1}^{\infty}\mathscr{H}_i$ be a closed cover of M that refines $\mathscr{U}$ such that, for each i, $\mathscr{H}_i$ is a discrete collection of sets. For each n, let $\mathscr{K}_n = \{f^{-1}(H) : H \in \mathscr{H}_n\}$ and let $\mathscr{K} = \bigcup_{n=1}^{\infty}\mathscr{K}_n$. Then for each n, $\mathscr{K}_n$ is a discrete collection of closed sets in X. Furthermore, if n is a positive integer and $K \in \mathscr{K}_n$, then K is not in $\mathscr{F}$.

We have two cases to consider:

Subcase 1. There is an integer n such that $\bigcup K_n$ is in $\mathscr{F}$: Let A be an indexing set for $\mathscr{H}_n$ so that we have $\mathscr{H}_n = \{H(a) : a \in A\}$. We will view A as a topological space with the discrete topology. For each a in A, $K(a)$ will denote the set $f^{-1}(H(a))$ so that A is also an indexing set for $\mathscr{K}_n$. If $B \subset A$, then let $K(B) = \bigcup_{a\in B}K(a)$ and $H(B) = \bigcup_{a\in B}H(a)$. Let $\mathscr{B} = \{B \subset A : K(B) \in \mathscr{F}\}$. It is clear that $\mathscr{B}$ has the finite intersection property. Furthermore, since it is true that if $B \subset A$, then either $K(B) \in \mathscr{F}$ or $K(A - B) \in \mathscr{F}$, it follows that $\mathscr{B}$ is an ultrafilter of subsets of A. Note that if $a \in A$, then $K(\{a\})$ is not in $\mathscr{F}$. Thus, $\mathscr{B}$ is a free ultrafilter in A. Since A has the same cardinality as a discrete subset of X, it follows that A is realcompact; and so, there is a countable subset $\{B_1, B_2, \ldots\}$ of $\mathscr{B}$ with $\bigcap B_i = \emptyset$. It follows that $\{K(B_1), K(B_2), \ldots\}$ is a countable subset of $\mathscr{F}$ with empty intersection and $\{H(B_1), H(B_2), \ldots\}$ is also a centered collection of closed subsets of M with empty intersection. It is true then that $\{M - H(B_1), M - H(B_2), \ldots\}$ is a countable open cover of X. Since M is subparacompact, M is countably metacompact; thus, there is a point finite open cover $\{U_1, U_2, \ldots\}$ of M such that, for each k, $U_k \subset (M - H(B_k))$. For each k, let $V_k = \bigcup_{j>k}U_j$. For each k, it follows that $\bigcap_{j=1}^{k}H(B_j) \subset f^{-1}(V_k)$. Since X is normal, there is, for each k, a zero-set Z_k such that $\bigcap_{j=1}^{k} K(B_j) \subset Z_k \subset f^{-1}(V_k)$. Thus, $\{Z_1, Z_2, \ldots\}$ is a countable subset of $\mathscr{Z}$ with $\bigcap Z_i = \emptyset$.

Subcase 2. For each n, it is true that $\bigcup \mathscr{K}_n$ is not in $\mathscr{F}$. In this case, for each n, let F_n be a member of $\mathscr{F}$ that does not intersect $\bigcup \mathscr{K}_n$. Since X is normal, there is, for each n, a zero-set Z_n such that $F_n \subset Z_n \subset X - \bigcup \mathscr{K}_n$. It follows that $\{Z_1, Z_2, \ldots\}$ is a countable subcollection of $\mathscr{Z}$ with empty intersection. This completes the proof that (2) implies (1).

(1) $\Rightarrow$ (3): This is a corollary to Lemma 1.

(3) $\Rightarrow$ (1): The argument for this implication is the same as that for Case 1 in the proof that (2) implies (1).

COROLLARY 1. *Every normal subparacompact T_1-space with property* (*) *is realcompact.*

COROLLARY 2. (Moran [7]). *Every normal metacompact T_1-space with property* (*) *is realcompact.*

In view of our Theorem, two questions seem to be of interest:

Question 1. Is every normal subparacompact T_1-space complete with respect to a uniformity on the space? Indeed, must every normal Moore space be complete with respect to some uniform structure?

Question 2. Must every metacompact normal T_1-space be complete with respect to some uniform structure on the space?

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DESCENDINGLY INCOMPLETE ULTRAFILTERS AND THE CARDINALITY OF ULTRAPOWERS

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Let D be an ultrafilter on I, and let κ be a cardinal. D is said to be *κ-descendingly incomplete* (κ-d.i.) if there exists a chain $X_\alpha : \alpha < \kappa$ of elements of D such that $\alpha < \beta \rightarrow X_\alpha \subseteq X_\beta$ and $X_\alpha = \phi$. Such a chain will be called a *κ-chain* for D. The notion of κ-descending incompleteness is due to Chang [**3**].

In this paper we explore the relationship between the cardinality of the ultrapower κ^I/D and the existence of certain chains on D. Since we deal so much with questions of size, we do not ordinarily make a notational distinction between a set and its cardinality. Where such a distinction is useful, the cardinality of a set A will be denoted by $|A|$.

The cardinal κ has a natural well-ordering which we denote by $<$. In the usual way, $<$ induces an order on κ^I/D, which we also denote by $<$. There is a natural (order-preserving) embedding of κ into κ^I/D. We make the usual identification and assume that $\kappa \subseteq \kappa^I/D$.

The following result is already implicit in Chang [**3**].

LEMMA 1. *κ is bounded above in κ^I/D with respect to $<$ if and only if D is κ-d.i.*

Proof. Suppose that κ is bounded in κ^I/D, and let $f/D \in \kappa^I/D$ be an upper bound for κ. For any $\alpha < \kappa$, let $X_\alpha = \{i \mid f(i) > \alpha\}$. It is clear that $\{X_\alpha : \alpha < \kappa\}$ is a κ-chain for D.

Conversely, let $\{X_\alpha : \alpha < \kappa\}$ be a κ-chain for D. Define $f : I \rightarrow \kappa$ by $f(i) = \alpha$ if and only if $i \in X_\alpha - X_{\alpha+1}$. Then f/D is an upper bound for κ in κ^I/D.

For ultrafilters D which are not κ-d.i., we obtain a representation for κ^I/D in terms of ultrapowers of smaller objects.

LEMMA 2. *If D is not κ-d.i., then*

$$|\kappa^I/D| = \sup_{\alpha<\kappa} |\alpha|^I/D.$$

Proof. For any $\alpha < \kappa$, let $C_\alpha = \{f/D \mid f/D < \alpha\}$. By Lemma 1, κ is confinal in κ^I/D with respect to $<$, and so we have the representation $\kappa^I/D = \bigcup_{\alpha<\kappa} C_\alpha$. But from the definition of C_α, $|C_\alpha| = |\alpha|^I/D$.

It is well-known that if D is regular and κ-d.i., then $\kappa^I/D > \kappa$. This is essentially a restatement of the fact that if $|I| = \kappa$ and D is uniform, then $\kappa^I/D > \kappa$. The main result of this paper is a partial converse of this theorem.

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If 2^κ is the nth successor of κ for some integer n, a converse essentially as strong as can be expected will follow.

Let $f : I \rightarrow A$. Put $i \backsim i'$ if $f(i) = f(i')$. The relation $\backsim$ partitions I. This partition will be called the partition induced by f. If Π is any partition of I, define $g : I \rightarrow \Pi$ by $g(i) = C$, where $C \in \Pi$ is the cell to which i belongs. Let D be an ultrafilter on I. We can now define an ultrafilter E on Π by putting $X \in E$ if and only if $g^{-1}(X) \in D$. E will be called the image of D on Π. In this situation, there is for any A a natural embedding of A^Π/E in A^I/D. For to any $f/E \in A^\Pi/E$ there corresponds an object $\hat{f}/D \in A^I/D$, where $\hat{f}(i) = f(C)$ for any cell C and any $i \in C$. A^Π/E will be treated as a subset of A^I/D.

For any ultrafilter D, let th(D) (the thickness of D) be the smallest cardinal μ such that there exists $X \in D$ with $|X| = \mu$. The following small observation will be needed in the proof of the main result:

LEMMA 3. *Let* $\Pi_\alpha : \alpha < \rho$ *be a sequence of partitions of* I *such that if* $\alpha < \beta$, Π_β *is a proper refinement of* Π_α. *Then for any ordinal* η, $|\Pi_\eta| \geqq |\eta|$.

Proof. We define a sequence C_α of subsets of I such that for any α, C_α meets a cell of Π_α in at most one point. Let $C_0 = \phi$. For any α, $C_{\alpha+1} = C_\alpha \cup \{p\}$ where p is in a cell of $\Pi_{\alpha+1}$ to which no element of C_α belongs. This is possible since $\Pi_{\alpha+1}$ is a proper refinement of Π_α. For α a limit ordinal,

$$C_\alpha = \bigcup_{\beta<\alpha} C_\beta.$$

Then clearly $|\Pi_\eta| \geqq |C_\eta| = |\eta|$.

We have now:

THEOREM 1. *Let* $\kappa^I/D > \kappa$. *Let* λ *be the smallest cardinal such that* $\kappa^\lambda > \kappa$. *Then* D *is* μ*-d.i. for some* μ *with* $\lambda \leqq \mu \leqq \max(\kappa, 2^\lambda)$.

Proof. If we can show that the ultrafilter D has an image E such that th$(E) = \mu$, then D is μ-d.i. For any non-principal ultrafilter E is th(E)-d.i., and since E is an image of D, from any μ-chain in E it is easy to construct a μ-chain in D.

Let $f_\alpha/D : \alpha < \kappa^+$ be a sequence of κ^+ distinct elements of κ^I/D. For each α, the partition induced by f_α has cardinality $\leqq\kappa$. Indeed without loss of generality we may assume that for each α the partition induced by f_α has cardinality $<\lambda$. For if the cardinality of the smallest partition induced by a representative of f/D is μ, then D has an image of thickness μ, and hence a μ-chain.

We now define a sequence Π_α of partitions of I. Let Π_0 be the one cell partition. For any α, let $\Pi_{\alpha+1}$ be the common refinement of Π_α and the partition induced by f_β, where f_β/D is the first element of our sequence which does not have a representative constant on the cells of Π_α. If α is a limit ordinal, let Π_α be the common refinement of all the Π_β with $\beta < \alpha$. For some ordinal $\eta \leqq \kappa^+$, every f_β/D has representative constant on the cells of Π_η, and the process of choosing the Π_α terminates. For $\alpha \leqq \eta$, let D_α be the image of D on Π_α. It is clear that (under our identification) $f_\alpha/D \in \kappa^{\Pi}\eta/D_\eta$ for all $\alpha < \kappa^+$. If $\eta \leqq \lambda$, we are through. For since $\Pi_{\alpha+1}$ divides any cell of Π_α into $<\lambda$ pieces, $|\Pi_\eta| \leqq 2^\lambda$. But since $\kappa^{\Pi}\eta/D_\eta \geqq \kappa^+$, $\mathrm{th}(D_\eta) \geqq \lambda$, and so

$$\lambda \leqq \mathrm{th}(D_\eta) \leqq |\Pi_\eta| \leqq 2^\lambda.$$

But then D is μ-d.i. for some μ with $\lambda \leqq \mu \leqq 2^\lambda$.

If $\eta \geqq \lambda$, we show that already $\mathrm{th}(D_\lambda) \geqq \lambda$. Since $\mathrm{th}(D_\lambda) \leqq 2^\lambda$, this will complete the argument. Let $\mathrm{th}(D_\lambda) = \rho$, and let $\Pi_\lambda{}^*$ be an element of D_λ of cardinality ρ. For any α we have a natural projection map $\phi_\alpha : \Pi_\lambda \rightarrow \Pi_\alpha$. Let $\Pi_\alpha{}^* = \phi_\alpha(\Pi_\lambda{}^*)$. We show that for any $\alpha < \lambda$, $\Pi^*_{\alpha+1}$ refines $\Pi_\alpha{}^*$ properly.

For suppose that $\Pi^*_{\alpha+1} = \Pi_\alpha{}^*$, and let f_β be any function constant on the cells of $\Pi_{\alpha+1}$. We define a function $g_\beta : I \rightarrow \kappa$. Let C be any cell of Π_α, and K be the collection of $i \in C$ which belong to some cell of $\Pi_\lambda{}^*$. Suppose there is some $i_0 \in K$. If $i \in C$, let $g_\beta(i) = f_\beta(i_0)$. If $K = \phi$, let g_β be constant on C.

Now if $i \in K$, since $\Pi^*_{\alpha+1} = \Pi_\alpha{}^*$, i and i_0 must belong to the same cell of $\Pi_{\alpha+1}$, and so $g_\beta(i) = f_\beta(i)$. So f_β and g_β agree on an element of D, and hence f_β/D has a respresentative constant on the cells of Π_α, contradicting the choice for $\Pi_{\alpha+1}$. It follows that $\Pi^*_{\alpha+1}$ is a proper refinement of $\Pi_\alpha{}^*$.

But now it follows immediately from Lemma 3 that $\mathrm{th}(D_\lambda) = |\Pi_\lambda{}^*| \geqq \lambda$, and so Theorem 1 is proved.

It seems plausible that the upper bound for μ obtained in Theorem 1 can be improved to κ. This would yield a best possible result, since for any μ, if D is a regular ultrafilter on μ, $|\kappa^\mu/D| = \kappa^\mu$. If κ and λ are as in the statement of Theorem 1, and $2^\lambda \leqq \kappa$, then Theorem 1 yields a best possible upper bound directly. Lemma 2 can be used together with Theorem 1 to deal with other rather special cardinals κ, but we have no generally valid argument that will improve our upper estimate in all cases.

If 2^κ is the nth successor of κ for some integer n, then the upper bound for μ can indeed be improved to κ. This is a routine application of the main result of [**6**]. So in particular we have:

COROLLARY 1 (G.C.H.). *Let κ be regular. If $\kappa^I/D > \kappa$, then D is κ-d.i.*

For κ singular, assuming the G.C.H., it is tempting to believe that if $\kappa^I/D > \kappa$, D is $\mathrm{cf}(\kappa)$-d.i. However, if we assume the existence of measurable cardinals, a counterexample can be exhibited using ideas similar to those of [**1**].

COROLLARY 2 (G.C.H.). *If κ is regular, and $\kappa^I/D > \kappa$, then $|(\kappa^I/D)^\kappa| = |\kappa^I/D|$.*

Proof. Chang [**3**] has shown that if

$$\kappa = \sum_{\delta<\gamma} \kappa^\delta,$$

and D is γ-d.i., then $|(\kappa^I/D)^\gamma| = |\kappa^I/D|$. By Corollary 1, Chang's condition is fulfilled with $\gamma = \kappa$. From Keisler's inequality $(\kappa^\lambda)^I/D \leqq (\kappa^I/D)^\lambda$ [**4**] one can only conclude that $\mathrm{cf}(|\kappa^I/D|) \geqq \kappa$.

COROLLARY 3 (C.G.H.). *Let κ be regular. Then κ is confinal in κ^I/D if and only if $|\kappa^I/D| = \kappa$.*

Proof. The proof follows by Lemma 1.

From the proof of Theorem 4, it is easy to see that (assuming 2^κ is the nth successor of κ for some n) if $|\kappa^I/D| = \kappa$, there is an ultrafilter E on a set J with $|J| < \kappa$ such that $\kappa^I/D \simeq \kappa^J/E$. So if we think of κ as being equipped with its full structure (all relations and functions on κ), κ^I/D is a simple extension of κ [**2**]. It is natural to ask here whether in the proof of this purely algebraic result special assumptions about exponentiation of cardinals can be eliminated. It is also reasonable to expect that if $|\kappa^I/D| \leqq 2^\kappa$, there is an ultrafilter E on a set J with $|J| \leqq \kappa$ such that $\kappa^I/D \simeq \kappa^J/E$. At this moment these questions remain open.

Define a function f from ordinals to cardinals by putting $f(0) = \omega$, $f(\alpha + 1) = |(f(\alpha))^I/D|$, and for limit ordinals β, $f(\beta) = \sup_{\alpha<\beta} f(\alpha)$. The function f reaches a maximum $\mu \leqq |2^I|$.

COROLLARY 4 (G.C.H.). *μ is the smallest cardinal such that D is not μ-d.i. In particular, if $|\omega^I/D| = |\omega^I|$, then D is κ-d.i. for all infinite $\kappa \leqq |I|$.*

In the proof of the next result, we need the fact that if $(2^\kappa)^I/D > 2^\kappa$, then $\kappa^I/D > \kappa$. Without any additional trouble we can prove the slightly stronger

LEMMA 4. $(A^B)^I/D \geqq (A^I/D)^{B^I/D}$.

Proof.† Any second order existential sentence true in a model $\mathscr{C}$ is true in every ultrapower of $\mathscr{C}$. Consider the model $\mathscr{C} = \langle A^B, A, B, R\rangle$ where $R(a, b, f)$ if and only if $f(b) = a$. In the model $\mathscr{C}^I/D$ we have

$$(A^B)^I/D \subseteq (A^I/D)^{B^I/D}$$

with the obvious identification induced by R^I/D.

Lemma 4 quickly yields that if $(2^\kappa)^I/D > 2^\kappa$, then $\kappa^I/D > \kappa$. For let $A = 2$, $B = \kappa$. Then $(2^\kappa)^I/D \leqq 2^{\kappa^I/D}$ and so if $(2^\kappa)^I/D > 2^\kappa$, we must have $\kappa^I/D > \kappa$.

Our final result gives a very weak estimate for the cardinality of κ^I/D when κ is a limit cardinal in terms of cardinalities of ultrapowers of cardinals smaller than κ.

THEOREM 2 (G.C.H.). *Let κ be a limit cardinal. Suppose there is a sequence λ_α of regular cardinals such that $\lambda_\alpha \to \kappa$ and $\lambda_\alpha{}^I/D > \lambda_\alpha$. Then $\kappa^I/D > \kappa$.*

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† We thank the referee for suggesting this simple proof.

Proof. By Corollary 1, D has a λ_α-chain for all α. From a chain $\{X_\beta : \beta < \lambda_\alpha\}$ one obtains a partition Π_α of I whose cells are the sets $X_{\beta+1} - X_\beta$. Since we have a λ_α-chain and λ_α is regular, the image E_α of D on Π_α is uniform. Let Π be the common refinement of the partitions Π_α, and let E be the image of D on Π. $|\Pi| \leqq 2^\kappa = \kappa^+$. Since each E_α is an image of E, and $\mathrm{th}(E_\alpha) = \lambda_\alpha$, $\mathrm{th}(E) \geqq \kappa$. If $\mathrm{th}(E) = \kappa$, E is κ-d.i., so $\kappa^I/D > \kappa$. If $\mathrm{th}(E) = \kappa^+ = 2^\kappa$, $(2^\kappa)^I/D > 2^\kappa$, and so $\kappa^I/D > \kappa$.

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MAXIMAL QUOTIENT RINGS AND S-RINGS

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Throughout, we assume all rings are associative with identity and all modules are unitary. See [**7**] for undefined terms and [**3**] for all homological concepts.

Let R be a ring, $E(R)$ the injective envelope of ${}_RR$, and $H = \mathrm{Hom}_R(E(R), E(R))$. Then we obtain a bimodule ${}_RE(R)_H$. Let $Q = \mathrm{Hom}_H(E(R), E(R))$. Q is called the *maximal left quotient ring of* R. Q has the property that if $p, q \in Q$, $p \neq 0$, then there exists $r \in R$ such that $rp \neq 0$, $rq \in R$, i.e., Q is a ring of left quotients of R.

A left ideal I of R is *dense* if for every $x, y \in R$, $x \neq 0$, there exists $r \in R$ such that $rx \neq 0$, $ry \in I$. An alternate description of Q is $Q = \{x \in E({}_RR) : (R : x)$ is a dense left ideal of $R\}$, where $(R : x) = \{r \in R : rx \in R\}$.

The *left singular ideal of* R is $Z_l(R) = \{r \in R : l_R(r)$ is an essential left ideal of $R\}$, where $l_R(r) = \{x \in R : xr = 0\}$. If $Z_l(R) = (0)$, then Q is a left self-injective von Neumann regular ring [**7**, § 4.5]. Most of the previous work on maximal left quotient rings has been done in this case. For example, Q is semisimple Artinian if and only if $Z_l(R) = (0)$ and R is finite-dimensional [**14**].

Our principal object is to study the maximal left quotient ring of a ring R whose left singular ideal is not necessarily zero. We begin in § 1 by considering right S-rings and, more generally, S-modules. The class of right S-rings contains all quasi-Frobenius and commutative perfect rings. We restrict our attention to maximal left quotient rings which are right S-rings.

In § 2, we show that the maximal left quotient ring Q of R is a semiprimary right S-ring if and only if (1) $QA = Q$ for every dense left ideal A of R, (2) dense left ideals can be lifted modulo $P(Q) \cap R$, and (3) $P(Q) \cap R$ is nilpotent, where $P(Q)$ denotes the prime radical of Q. Rings R for which Q is a left Artinian right S-ring are also characterized, and these results are applied to local rings. Various descriptions of the ideal $P(Q) \cap R$ are obtained.

In § 3, we consider maximal left quotient rings which are left self-injective semiprimary and quasi-Frobenius rings. We also show that if R has a left Artinian classical left quotient ring Q, then Q is a left and right S-ring if and only if $l_Rr_R(P(R)) = P(R)$ and $r_Rl_R(P(R)) = P(R)$. This result is applied to quasi-Frobenius rings.

1. S-rings. Let ${}_R\mathscr{M}$ denote the category of left R-modules. Then $\mathscr{T} = \{M \in {}_R\mathscr{M} : \mathrm{Hom}_R(M, E({}_RR)) = (0)\}$ is a hereditary torsion class, i.e.,

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$\mathcal{T}$ is closed under homomorphic images, arbitrary direct sums, extensions, and submodules. The filter associated with $\mathcal{T}$ is $\mathcal{F} = \{I : I$ is a left ideal of R such that $R/I \in \mathcal{T}\} = \{I : I$ is a dense left ideal of $R\}$. $\mathcal{F}$ has the following properties:

(1) If $L \in \mathcal{F}$ and $L \subseteq L' \subseteq R$, then $L' \in \mathcal{F}$.
(2) If $L, L' \in \mathcal{F}$, then $L \cap L' \in \mathcal{F}$.
(3) If L $\in \mathcal{F}$ and $r \in R$, then $(L : r) \in \mathcal{F}$.
(4) If $L, L' \in \mathcal{F}$, then $LL' \in \mathcal{F}$.

For a proof, see [**1**, Proposition 1.2.8].

If $M \in {}_R\mathcal{M}$, let $T(M)$ denote the $\mathcal{T}$-torsion submodule of M, i.e., $T(M)$ is the (unique) submodule of M maximal in the set of submodules A of M for which $\operatorname{Hom}_R(A, E({}_RR)) = (0)$. Then the maximal left quotient ring Q of R is given by $Q = T(E({}_RR)/R) = \{x \in E({}_RR) : (R : x)$ is a dense left ideal of $R\}$ (see [**18**]). We will use this characterization of Q extensively.

The dense left ideals of R and Q are related in the following way.

Lemma 1.1. *Let Q be the maximal left quotient ring of R, and let T be a subring of Q such that $R \subseteq T \subseteq Q$. Then:*

(1) *If A is a dense left ideal of R, then TA is a dense left ideal of T.*
(2) *If B is a dense left ideal of T, then $B \cap R$ is a dense left ideal of R.*

Before proceeding with our study of quotient rings, we will introduce S-modules. In particular, we will consider S-rings and their relationship to dense left ideals and maximal quotient rings.

Definition. ${}_RM$ is an S-module if M contains a copy of each of its simple images.

Proposition 1.2. *Let M be a left R-module and $E = \operatorname{End}_R(M)$. Consider the following conditions:*

(1) *M is an S-module;*
(2) *for every submodule $N \neq M$, $r_E(N) \neq (0)$;*
(3) *for every essential submodule $N \neq M$, $r_E(N) \neq (0)$;*
(4) *for every maximal submodule N of M, $r_E(N) \neq (0)$.*

Then (1) *if and only if* (4), (2) *if and only if* (3), *and* (2) *implies* (4). *Furthermore, if M is finitely generated, then* (4) *implies* (2), *so all four conditions are equivalent.*

Proof. Trivially, (2) implies (3) and (4). That (1) and (4) are equivalent follows easily from the fact that a submodule A of M is maximal if and only if M/A is simple. Assume (3) holds, and let N be a proper submodule of M. Choose $K \subseteq M$ maximal with respect to $N \cap K = (0)$. Then $N + K$ is an essential submodule of M. If $N + K \neq M$, then $r_E(N) \supseteq r_E(N + K) \neq (0)$ by (3). If $N + K = M$, then $M = N \oplus K$, and there exists $0 \neq f \in E$ defined by $f(n, k) = k$, where $n \in N$, $k \in K$. Thus (3) implies (2). If M is finitely generated, then every proper submodule of M is contained in a maximal submodule of M, so (4) implies (2).

Remark. An arbitrary direct sum of S-modules is an S-module, but a direct summand of an S-module need not be an S-module.

PROPOSITION 1.3. *If ${}_RM$ is a finitely generated S-module such that $Z(M) = (0)$, then M is completely reducible.*

Proof. Let N be a maximal submodule of M. Then M contains a copy of M/N, so $Z(M/N) = (0)$. Assume N is an essential submodule of M. If $0 \neq m' \in M/N$, then $(N : m)$ is essential, so $0 \neq m' \in Z(M/N)$. But this is a contradiction, so N is not essential. If A is a proper essential submodule of M, then $A \subseteq K$ for some maximal submodule K of M since M is finitely generated. Then K is essential since A is essential, but we proved above that no maximal submodule of M is essential. Thus M has no proper essential submodules, so M is completely reducible.

PROPOSITION 1.4. *Let R be a semiprime ring. If ${}_RM$ is a finitely generated projective S-module, then M is completely reducible.*

Proof. Let N be a maximal submodule of M. Since M is an S-module, M contains a copy of M/N. Also M can be embedded in a finite direct sum of copies of R since M is finitely generated projective. Thus there exists a homomorphism $0 \neq f \in \operatorname{Hom}_R(M/N, R)$. Since M/N is simple, $I = f(M/N)$ is a minimal left ideal of R. Then $I = Re$, e an idempotent in R, since R is semiprime. Thus $M/N \cong f(M/N) = Re$ is projective, so N is a direct summand of M. Hence every maximal submodule of M is a direct summand. Thus M is completely reducible as in the previous proposition.

Definition. ${}_RM$ is *semiprime* if for every $0 \neq m \in M$, there exists $f \in \operatorname{Hom}_R(M, R)$ such that $f(m)m \neq 0$.

PROPOSITION 1.5. *If ${}_RM$ is a finitely generated semiprime S-module, then M is completely reducible.*

Proof. Let A be a maximal submodule of M. There exists a submodule K of M such that $K \cong M/A$. Let $0 \neq x \in K$. Then $K = Rx$ since K is simple. Since M is semiprime, there exists $f \in \operatorname{Hom}_R(M, R)$ such that $f(x)x \neq 0$. Then $f(K) \neq (0)$, so $f(K)$ is a minimal left ideal of R. $K = Rf(x)x$, so $(0) \neq f(K) = f(Rf(x)x) = Rf(x)^2$. Hence $f(x)^2 \neq 0$, so $f(K)^2 \neq (0)$. Therefore, $M/A \cong K \cong f(K) = Re$, e an idempotent of R. Now proceed as in the proof of the previous proposition.

Definition. R is a *right* (*left*) *S-ring* if each proper left (right) ideal of R has nonzero right (left) annihilator.

PROPOSITION 1.6. *The following are equivalent.*

(1) *R is a right S-ring.*
(2) *R contains a copy of each simple left R-module.*
(3) *R has no proper dense left ideals.*

Proof. See [**6**, Theorem 3.2].

COROLLARY. *R is a right S-ring if and only if ${}_RR$ is an S-module.*

PROPOSITION 1.7. *Let R be a commutative ring. Then R is a right S-ring if and only if R has a faithful finitely generated projective S-module.*

Proof. If R is a right S-ring, then ${}_RR$ is a faithful finitely generated projective S-module. Assume R has a faithful finitely generated projective S-module ${}_RM$. Then M is a generator in the category of R-modules [**20**, Proposition 1.3], so every simple R-module is a homomorphic image of M. Let K be a simple R-module. Since M is an S-module and K is a homomorphic image of M, there exists a submodule L of M such that $L \cong K$. Also there exists $f \in \operatorname{Hom}_R(M, R)$ such that $f(L) \neq (0)$ since M is finitely generated projective. Thus R contains a copy of every simple R-module, so R is a right S-ring.

PROPOSITION 1.8. *If R is a right S-ring which satisfies the descending chain condition on left annihilators, then $R/J(R)$ is semisimple Artinian.*

Proof. Since R is a right S-ring, every simple left R-module is isomorphic to a minimal left ideal of R. Hence $J(R) = l_R(T)$, where T is the left socle of R. But R satisfies the descending chain condition on left annihilators, so there exist $x_1, \ldots, x_n \in T$ such that $J(R) = l_R(x_1) \cap \ldots \cap l_R(x_n)$. Then $Rx_1 \oplus \ldots \oplus Rx_n$ is a finite direct sum of simple left R-modules, and there exists a monomorphism from $R/J(R)$ into $Rx_1 \oplus \ldots \oplus Rx_n$. Thus $R/J(R)$ is semisimple Artinian.

Definition. Let I be a two-sided ideal of R. We say that dense left ideals can be *lifted modulo I* if whenever A is a left ideal of R such that $(A + I)/I$ is a dense left ideal of R/I, then A is a dense left ideal of R.

PROPOSITION 1.9. *Let R be a left Noetherian right S-ring. The following are equivalent:*

(1) *R is left Artinian.*
(2) *R satisfies the regularity condition.*
(3) *Dense left ideals can be lifted modulo $P(R)$.*
(4) *R satisfies the descending chain condition on left annihilators and $J(R)$ is nilpotent.*

Proof. (1) implies (2) by Small's theorem [**15**; **16**]. (4) implies (1) by Proposition 1.8, and clearly (1) implies (4). R is its own maximal left quotient ring Q since it is a right S-ring. Hence $P(Q) \cap R = P(R)$. Thus (3) implies (1) by Theorem 2.9 (see § 2). It remains only to show that (2) implies (3). Suppose I is a left ideal of R such that $(I + P(R))/P(R)$ is a dense left ideal of $R/P(R)$. Then $(I + P(R))/P(R)$ contains a regular element $a + P(R)$ since $R/P(R)$ is semiprime left Goldie [**12**, Lemma 5]. By Small's theorem, R has a classical left quotient ring Q. Then $1 = a^{-1}a \in Q(I + P(R))$. Suppose $x \in (R : a^{-1})$.

Then $x = x \cdot 1 = xa^{-1}a \in Ra \subseteq I + P(R)$, so $(R : a^{-1}) \subseteq I + P(R)$. Hence $I + P(R)$ is a dense left ideal of R, so $I + P(R) = R$ since R is a right S-ring. This implies that $I = R$ since $P(R)$ is small in R, so I is a dense left ideal of R. Thus (2) implies (3).

The following theorem is known [**19**, Theorem 3.2].

THEOREM 1.10. *Let Q be the maximal left quotient ring of R. The following are equivalent:*

(1) *Q is a right S-ring.*
(2) *$QA = Q$ for every dense left ideal A of R.*
(3) $\operatorname{Ker}(R \otimes {}_R M \to Q \otimes {}_R M) = T({}_R M)$ *for every left R-module M.*

COROLLARY. *Suppose the maximal left quotient ring Q of R is a right S-ring. If A is a flat left R-module, then* $T({}_R A) = (0)$.

COROLLARY. *If the maximal left quotient ring Q of R is a right S-ring, then* $\text{l.gl.dim.}R \geqq \text{l.gl.dim.}Q$.

Proof. Let M be a left Q-module, and let $\cdots \to P_2 \to P_1 \to P_0 \to M \to (0)$ be a projective resolution of M as a left R-module. Then because $T(P_i) = (0)$ and Q is a right S-ring, $\cdots \to Q \otimes P_2 \to Q \otimes P_1 \to Q \otimes P_0 \to Q \otimes M \to (0)$ is a projective resolution of $M \cong Q \otimes M$ as a left Q-module. Hence $\operatorname{pd}({}_Q M) \leqq \operatorname{pd}({}_R M)$. The result follows.

Example. Let $0 \leqq n \leqq \infty$. Then there exists a ring R with maximal left quotient ring Q such that $\text{l.gl.dim.}R = n$ and $\text{l.gl.dim.}Q = 0$.

Case 1: $n = 0$. Let R be any semisimple Artinian ring. Then $R = Q$, and $\text{l.gl.dim.}R = 0$.

Case 2: $0 < n < \infty$. Let K be a field and $R = K[x_1, \ldots, x_n]$. Then $\text{gl.dim.}R = n$ [**3**, IX, 7.11]. R is a commutative domain, so R has a classical quotient ring Q which is a field. Then Q is the maximal quotient ring of R, and $\text{gl.dim.}Q = 0$.

Case 3: $n = \infty$. Let K be a field, and let R be the subring of $K[[x]]$ consisting of all power series without terms of degree 1. R is a local ring [**3**, Exercise 10, p. 160]. Also R is a commutative domain, so it has a classical quotient ring Q which is a field. Then Q is the maximal quotient ring of R, and $\text{gl.dim.}Q = 0$. Let A be the R-module consisting of all power series without a constant term. Then $\operatorname{pd}_R A = \infty$ [**3**, Exercise 10, p. 160]. Hence $\text{gl.dim.}R = \infty$.

If R has a minimal dense left ideal D, then the maximal left quotient ring Q of R is isomorphic to $\operatorname{Hom}_R(D, D)$ [**7**, Corollary 3, p. 97]. For example, if R is right perfect, then R has a minimal dense left ideal, and this case has been studied by Storrer [**17**]. In particular, Storrer obtained the following result.

PROPOSITION 1.11. *Let R be a ring which has a minimal dense left ideal D. Then the maximal left quotient ring Q of R is a right S-ring if and only if ${}_R D$ is finitely generated projective.*

Proof. See [**17**, Theorem 5.6].

Proposition 1.11 has immediate consequences when combined with some results of Mares and Ware. The following definition is due to Mares.

Definition. Let P be a projective R-module. P is a *semiperfect* module if every homomorphic image of P has a projective cover. P is *perfect* if for every index set I, every homomorphic image of $\oplus \sum_{i \in I} P$ has a projective cover.

PROPOSITION 1.12. *Let R be a ring which has a minimal dense left ideal D. Then the maximal left quotient ring Q of R is a semiperfect (left perfect) right S-ring if and only if ${}_RD$ is a finitely generated projective semiperfect (perfect) module.*

Proof. The proof follows immediately from Proposition 1.11, Proposition 5.1 of [**20**] and its converse due to Mares [**9**], and the fact that $Q = \mathrm{Hom}_R(D, D)$.

Definition [**20**]. Let P be a nonzero projective R-module. P is *local* if P has a unique maximal submodule which contains every proper submodule of P.

PROPOSITION 1.13. *Let R be a ring which has a minimal dense left ideal D. The maximal left quotient ring Q of R is a local right S-ring if and only if D is a finitely-generated projective local R-module.*

Proof. The proof follows from Proposition 1.11 and [**20**, Theorem 4.2].

Definition [**20**]. Let P be a projective R-module. P is *regular* if every cyclic submodule of P is a direct summand.

PROPOSITION 1.14. *Let R be a commutative ring which has a minimal dense ideal D. The maximal quotient ring Q of R is a finite direct sum of fields if and only if ${}_RD$ is a finitely generated projective regular module.*

Proof. $Q = \mathrm{Hom}_R(D, D)$ is a regular S-ring if and only if ${}_RD$ is a finitely generated projective regular module by Proposition 1.11 and [**20**, Corollary 3.10]. If Q is a finite direct sum of fields, clearly Q is a regular S-ring. If Q is a regular S-ring, then $Z_l(Q) = (0)$, so Q is semisimple Artinian by Proposition 1.3. Since Q is also commutative, Q is a finite direct sum of fields.

Let R be a ring which has a classical left quotient ring Q. If I is a left ideal of Q, then $I = Q(I \cap R)$. This property does not hold in general for maximal left quotient rings, even if they are left Artinian (see [**17**, Example 7.2]). However, this problem is eliminated when the maximal left quotient ring is a right S-ring.

LEMMA 1.15. *Suppose the maximal left quotient ring Q of R is a right S-ring, and let J be a left ideal of Q. Then $J = Q(J \cap R)$.*

Proof. Clearly, $Q(J \cap R) \subseteq QJ \subseteq J$. Now let $x \in J$. $(R : x)$ is a dense left ideal of R, so $Q(R : x) = Q$ since Q is a right S-ring. Hence $x \in Qx =$

$Q(R:x)x = Q(Rx \cap R) \subseteq Q(J \cap R)$. Therefore $J \subseteq Q(J \cap R)$, so we have equality.

COROLLARY. *Suppose the maximal left quotient ring Q of R is a right S-ring, and let J be a two-sided ideal of Q. Then $J^k = Q(J \cap R)^k$ for $k = 1, 2, 3, \ldots$.*

Proof. We observe that

$$J^k = J^{k-1}Q(J \cap R) = J^{k-1}(J \cap R) = J^{k-2}Q(J \cap R)^2 = \ldots$$
$$= J(J \cap R)^{k-1} = Q(J \cap R)^k.$$

LEMMA 1.16. *Let R be a ring whose maximal left quotient ring Q is a right S-ring, and let J be a two-sided ideal of Q. Then:*

(1) *$(R+J)/J$ is an essential left $((R+J)/J)$-submodule of Q/J.*

(2) *If A/J is an essential left ideal of Q/J, then $(A/J) \cap (R+J)/J$ is an essential left ideal of $(R+J)/J$.*

(3) *If B is a left ideal of R such that $(B+J)/J$ is an essential left ideal of $(R+J)/J$, then $(Q/J)\cdot(B+J)/J$ is an essential left ideal of Q/J.*

(4) *Q/J is a ring of left quotients of $(R+J)/J$.*

(5) *Let $\{A_i : i \in I\}$ be a collection of left ideals of R. Then $Q(\sum_{i\in I} A_i) = \sum_{i\in I} QA_i$.*

Proof. We will prove only statements (1) and (5) since the proofs of (2)–(4) use similar techniques. Let $x \in Q/J$. Then $Q(R:x) = Q$ since Q is a right S-ring. Suppose $Rx \cap R \subseteq J$. Then $x \in Qx = Q(R:x)x = Q(Rx \cap R) \subseteq QJ = J$, which is a contradiction. Hence $Rx \cap R \not\subseteq J$. Thus (1) holds. Now let $\{A_i : i \in I\}$ be a collection of left ideals of R. Clearly $Q(\sum A_i) \subseteq \sum QA_i$. Let $x \in \sum QA_i$. Then $x = \sum q_j a_j$, $a_j \in A_{i_j}$ for some $i_j \in I$. Let $B = \bigcap (R : q_j)$. B is a dense left ideal of R, so $QB = Q$. Hence there exist $p_k \in Q$, $b_k \in B$ such that $1 = \sum p_k b_k$. Then $x = 1 \cdot x = \sum p_k b_k q_j a_j \in Q(\sum A_i)$. Thus $\sum QA_i \subseteq Q(\sum A_i)$, so we have equality, and (5) holds.

2. Artinian maximal quotient rings which are S-rings.

PROPOSITION 2.1. *Let Q be the maximal left quotient ring of R, and let $N = P(Q) \cap R$. Q is a right S-ring such that $Q/P(Q)$ is semisimple Artinian if and only if*

(1) *$QA = Q$ for every dense left ideal A of R, and*

(2) *dense left ideals can be lifted modulo N.*

Proof. Let us first assume that R satisfies conditions (1) and (2). Q is a right S-ring by (1). We claim that $Q/P(Q)$ is a right S-ring. Suppose I/P is a dense left ideal of Q/P, where $P = P(Q)$. Then $(I \cap R)/N$ is a dense left ideal of R/N, so $I \cap R$ is a dense left ideal of R by (2). Hence $I = Q(I \cap R) = Q$ by (1). Thus Q/P has no proper dense left ideals, so Q/P is a right S-ring. Then Q/P is semisimple Artinian by Proposition 1.4. Conversely, assume Q is a right S-ring such that Q/P is semisimple Artinian. R satisfies (1) since Q

is a right S-ring. Now let A be a left ideal of R such that $(A + N)/N$ is a dense left ideal of R/N. Then $(Q/P)((A + P)/P) = (QA + P)/P$ is an essential left ideal of Q/P, so $Q/P = \mathrm{socle}_l(Q/P) \subseteq (QA + P)/P$. Hence $QA + P = Q$, so $QA = Q$. Let $\sum q_i a_i + A \in (QA \cap R)/A$. Then $(A : \sum q_i a_i)$ is a dense left ideal of R, so $R(\sum q_i a_i + A) \cong R/(A : \sum q_i a_i) \in \mathscr{T}$. Hence $\mathrm{Hom}_R(C, E(R)) = (0)$ for every cyclic submodule C of $(QA \cap R)/A$, so $\mathrm{Hom}_R((QA \cap R)/A, E(R)) = (0)$. Therefore, $R/A = (QA \cap R)/A \in \mathscr{T}$, so A is a dense left ideal of R. Consequently, R satisfies (2). This completes the proof.

THEOREM 2.2. *Let Q be the maximal left quotient ring of R, and let $N = P(Q) \cap R$. Q is a semiprimary right S-ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R,*

(2) *dense left ideals can be lifted modulo N, and*

(3) *N is nilpotent.*

Proof. Assume R satisfies conditions (1)–(3). By Proposition 2.1, it suffices to show that $P(Q)$ is nilpotent. But this follows from (3) and the corollary to Lemma 1.15. The other half of the theorem follows immediately from Proposition 2.1.

THEOREM 2.3. *Let Q be the maximal left quotient ring of R, and let $N_k = QN^k \cap R$, where $N = P(Q) \cap R$. Q is a left Artinian right S-ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R,*

(2) *dense left ideals can be lifted modulo N,*

(3) *N is nilpotent, and*

(4) *R/N_k is finite-dimensional* ($k = 1, 2, 3, \ldots$).

Proof. Suppose R satisfies conditions (1)–(4). Then Q is a semiprimary right S-ring by Theorem 2.2. $R/N_k = R/(QN^k \cap R) = R/(P(Q)^k \cap R) \cong (R + P(Q)^k)/P(Q)^k$. Denote $P(Q) = J(Q)$ by J. $(R + J^k)/J^k$ is essential in Q/J^k by Lemma 1.16, so Q/J^k is finite-dimensional by (4). $J^{k-1}/J^k \subseteq \mathrm{socle}_l(Q/J^k)$, which is a finite direct sum of simples. Hence J^{k-1}/J^k is either a finite direct sum of simples or zero. Since Q is semiprimary, we get a composition series for Q, so Q is left Artinian. Conversely, if Q is a left Artinian right S-ring, then R satisfies (1)–(3) by Theorem 2.2. Q/J^k is a ring of left quotients of $(R + J^k)/J^k$ by Lemma 1.16, so Q/J^k can be embedded in the maximal left quotient ring of $(R + J^k)/J^k$. Therefore, $R/N_k \cong (R + J^k)/J^k$ is finite-dimensional since Q/J^k is finite-dimensional. Hence R satisfies condition (4), and the proof is complete.

THEOREM 2.4. *The maximal left quotient ring of R is a semiprimary* (*left Artinian*) *right S-ring if and only if the maximal left quotient ring of R_n is a semiprimary* (*left Artinian*) *right S-ring for every $n \geqq 1$.*

Proof. Let Q be the maximal left quotient ring of R. Half of the theorem is trivial, so let us suppose that Q is a semiprimary (left Artinian) right S-ring.

Then Q_n is the maximal left quotient ring of R_n, and Q_n is semiprimary (left Artinian). Then Q_n has a minimal dense left ideal D, and D is a two-sided ideal [**17**, Proposition 1.1]. There exists an ideal J of Q such that $D = J_n$ [**10**, Theorem 2.24]. $r_Q(J) = (0)$ since $r_{Q_n}(J_n) = (0)$. Hence J is a dense left ideal of Q, so $J = Q$ since Q is a right S-ring. Thus $D = J_n = Q_n$, so Q_n is a right S-ring.

Example. Rosenberg and Zelinsky [**13**, p. 375] have given a ring R which has only the left ideals (0), N, and R, where $N \neq (0)$ and $N^2 = (0)$. R is a left Artinian right S-ring which is not right Artinian.

Example. Let F be a field, and let

$$R = \left\{ \begin{vmatrix} a & 0 & 0 \\ b & a & 0 \\ d & 0 & e \end{vmatrix} : a, b, d, e \in F \right\}.$$

R is a left and right Artinian ring which is a right, but not a left, S-ring [**17**, Example 7.2].

Example. Let F be a field, and let

$$R = \begin{vmatrix} F & F \\ 0 & F \end{vmatrix}.$$

Then R is left and right Artinian, and $Q = F_2$ is the maximal left and maximal right quotient ring of R [**4**]. Hence R is neither a left nor a right S-ring (otherwise it would be left or right rationally complete). R is also a left and right QF-3 ring by [**5**, Theorem 5]. However, R is not quasi-Frobenius for then it would be both a left and a right S-ring.

We will now specialize some of the above results to the case where Q is a local ring. This case is of particular interest for then Q is the classical left quotient ring of R.

PROPOSITION 2.5. *Let Q be the maximal left quotient ring of R, and let $N = P(Q) \cap R$. Q is local with nilpotent radical if and only if*

(1) *$QA = Q$ for every dense left ideal A of R,*
(2) *dense left ideals can be lifted modulo N,*
(3) *N is nilpotent, and*
(4) *R/N is a finite-dimensional domain.*

In this case, Q is the classical left quotient ring of R.

Proof. Suppose R satisfies (1)–(4). Then Q is a semiprimary right S-ring by Theorem 2.2. Let $P = P(Q)$. Since $R/N \cong (R + P)/P$ is a finite-dimensional domain, it has a classical left quotient ring Q' which is a division ring. Then Q' is the maximal left quotient ring of $(R + P)/P$. Q/P is a ring of left quotients of $(R + P)/P$, so we can assume that $Q/P \subseteq Q'$. Every nonzero element of $(R + P)/P$ is invertible in Q/P, so $Q/P = Q'$. Hence Q/P is a division ring, so Q is a local ring. Also $J(Q) = P(Q)$ is nilpotent since Q is semiprimary. Conversely, suppose Q is local with nilpotent radical. Then Q is a semiprimary right S-ring, so R satisfies (1)–(3) by Theorem 2.2. $R/N \cong (R+P)/P \subseteq Q/P$ and Q/P is a division ring, so R/N is a domain. Also Q/P is a ring of left quotients of $(R+P)/P$ and Q/P is finite-dimensional, so $R/N \cong (R+P)/P$ is finite-dimensional. Thus R satisfies (4).

Let us now assume that R satisfies the conditions and show that Q is the classical left quotient ring of R. By the above proof, Q/P is the classical left quotient ring of $(R+P)/P$. Let $x \in R$ be regular. Then $x \notin P$ since P is nilpotent. Therefore, $x + P$ is invertible in Q/P, so there exists $q \in Q$ such that $xq - 1 \in P$ and $qx - 1 \in P$. But this implies that xq and qx are invertible in Q, so x is invertible in Q. Thus every regular element of R is invertible in Q. Let $q \in Q$. Then $(R : q)$ is a dense left ideal of R, so $Q(R : q) = Q$. Hence $(R : q) \nsubseteq P$, so there exists a unit $u \in Q$ such that $u \in (R : q)$ since Q is a local ring. Then u is a regular element of R and $q = u^{-1}r$ for some $r \in R$. Thus Q is the classical left quotient ring of R. This completes the proof.

COROLLARY. *Let Q be the maximal left quotient ring of R and let $N_k = QN^k \cap R$, where $N = P(Q) \cap R$. Q is a local left Artinian ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R,*
(2) *dense left ideals can be lifted modulo N,*
(3) *N is nilpotent,*
(4) *R/N_k is finite-dimensional ($k = 1, 2, 3, \ldots$), and*
(5) *R/N_1 is a domain.*

In this case, Q is the classical left quotient ring of R.

Proof. The proof follows immediately from Theorem 2.3 and Proposition 2.5.

PROPOSITION 2.6. *Let Q be the maximal left quotient ring of R, and let $N = P(Q) \cap R$. Q is a local, left Artinian, principal left ideal ring if the following conditions are satisfied:*

(1) *$QA = Q$ for every dense left ideal A of R.*
(2) *Dense left ideals can be lifted modulo N.*
(3) *R/N is a finite-dimensional domain.*
(4) *$N = Rx$ for some $x \in R$.*

Proof. N is nil, so x is nilpotent. Also $N^k = Rx^k$ for every positive integer k, so N is nilpotent. Therefore, Q is local with nilpotent radical by Proposition 2.5. $J(Q) = P(Q) = QN = QRx = Qx$, so every left ideal of Q is of the form $J(Q)^k = Qx^k$ by [**11**, Proposition 2.1]. Hence Q is a principal left ideal ring. Thus Q is left Noetherian and semiprimary, so Q is left Artinian.

In general, the ideal $N = P(Q) \cap R$ need not be the prime radical of R. We will now give various descriptions of this ideal. The following notation will be used:

$S_1 = \{A : A$ is a two-sided ideal of R, $A \subseteq P(R)$, and $AQ \subseteq QA\}$.
$S_2 = \{A : A$ is a left ideal of R and QA is nilpotent$\}$.
$S_3 = \{A : A$ is a two-sided ideal of R, $A \subseteq P(R)$, and A is the right annihilator of a subset of $Q\}$.

PROPOSITION 2.7. *Assume R is left Goldie and the maximal left quotient ring Q of R is a right S-ring. (Note that these conditions are satisfied if Q is a left Artinian right S-ring.) Then $N = P(Q) \cap R$ can be described in the following ways:*

(1) $N = \sum \{A : A \in S_1\}$.
(2) $N = \sum \{A : A \in S_2\}$.
(3) $N = \sum \{l_R(Q/QA) : A \in S_1\}$.

Also,

(4) $N \supseteq \sum \{A : A \in S_3\}$ *with equality if $P(Q)$ is a right annihilator in Q, and*
(5) $P(Q)^n \cap R = \sum \{QA^n \cap R : A \in S_1\} = QN^n \cap R$.

Proof. (1) Let $A \in S_1$. Then A is nilpotent since A is nil and R is left Goldie. Hence QA is nilpotent since $AQ \subseteq QA$. Therefore,

$$A \subseteq QA \cap R \subseteq P(Q) \cap R = N.$$

Thus $\sum \{A : A \in S_1\} \subseteq N$. But $N \in S_1$, so we have equality.

(2) If $A \in S_2$, then $QA \subseteq P(Q)$, so $A \subseteq N$. Hence $\sum \{A : A \in S_2\} \subseteq N$. Also $N \in S_2$.

(3) Let $A \in S_1$, and let $B = l_R(Q/QA)$. Then $BQ \subseteq QA$, so

$$B \subseteq BQ \cap R \subseteq QA \cap R \subseteq QN \cap R = P(Q) \cap R = N.$$

Hence $N \supseteq \sum \{l_R(Q/QA) : A \in S_1\}$. Now let $x \in N$. By (1) there exist $A_1, \ldots, A_n \in S_1$ such that $x \in A_1 + \ldots + A_n$. But $(A_1 + \ldots + A_n)Q \subseteq A_1Q + \ldots + A_nQ \subseteq QA_1 + \ldots + QA_n = Q(A_1 + \ldots + A_n)$ by Lemma 1.16. Hence $A_1 + \ldots + A_n \in S_1$, so $x \in A$ for some $A \in S_1$. Then

$$x(Q/QA) \subseteq A(Q/QA) = (0)$$

since $AQ \subseteq QA$. Therefore, $x \in l_R(Q/QA)$, so $N \subseteq \sum \{l_R(Q/QA) : A \in S_1\}$.

(4) Let $A \in S_3$, say $A = r_R(X)$, where $X \subseteq Q$. Then

$$A \subseteq AQ \cap R \subseteq r_R(X) = A,$$

so $A = AQ \cap R$. Suppose $\sum x_i q_i \in AQ$, $x_i \in A$, $q_i \in Q$. $B = \bigcap (R : x_i q_i)$ is a dense left ideal of R, so $QB = Q$ since Q is a right S-ring. Hence $1 = \sum p_j b_j$, $p_j \in Q$, $b_j \in B$. Now $\sum_i x_i q_i = \sum_{i,j} p_j b_j x_i q_i \in Q(AQ \cap R) = QA$. Hence $AQ \subseteq QA$, so $A \in S_1$. Hence $N \supseteq \sum \{A : A \in S_3\}$ by (1). If $P(Q)$ is a right annihilator in Q, then $N \in S_3$, so we have equality.

(5) $P(Q)^n \cap R = (QN)^n \cap R = QN^n \cap R$ by the corollary to Lemma 1.15. Now use (1).

In the remainder of this section, we will reconsider the results obtained in the first part of the section, this time restricting ourselves to certain classes of rings R. We begin by considering commutative rings.

THEOREM 2.8. *Let R be a commutative ring, Q the maximal quotient ring of R. Q is semiprimary if and only if*

(1) *$QA = Q$ for every dense ideal A of R,*

(2) *dense ideals can be lifted modulo $P(R)$, and*

(3) *$P(R)$ is nilpotent.*

Proof. First note that $P(R) = P(Q) \cap R$ since in a commutative ring the prime radical is just the set of nilpotent elements. Also a commutative semiprimary ring is a right S-ring. Hence the theorem follows immediately from Theorem 2.2.

THEOREM 2.9. *Let R be a left Noetherian ring. Then the maximal left quotient ring Q of R is a left Artinian right S-ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R, and*

(2) *dense left ideals can be lifted modulo $N = P(Q) \cap R$.*

Proof. Since R is left Noetherian, $R/(QN^k \cap R)$ is finite-dimensional for all k. Also N is nilpotent since N is nil and R is left Noetherian. The theorem now follows from Theorem 2.3.

PROPOSITION 2.10. *Let R be a right Noetherian ring. Then the maximal left quotient ring Q of R is a semiprimary right S-ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R, and*

(2) *$J(Q) \cap R$ is nilpotent.*

Proof. If Q is a semiprimary right S-ring, then R clearly satisfies (1) and (2). Now suppose conditions (1) and (2) are satisfied. Q is a right S-ring by (1). Also $J(Q)$ is nilpotent by (2) and the corollary to Lemma 1.15. Hence it suffices to show that $Q/J(Q)$ is semisimple Artinian. By Proposition 1.8, we need only show that Q satisfies the descending chain condition on left annihilators. Suppose $l_Q(X_1) \supseteq l_Q(X_2) \supseteq \ldots$. Intersecting each term of the chain with R we obtain the chain $l_R(X_1) \supseteq l_R(X_2) \supseteq \ldots$. But then $r_R l_R(X_1) \subseteq r_R l_R(X_2) \subseteq \ldots$, so there exists an n such that $r_R l_R(X_n) = r_R l_R(X_{n+1}) = \ldots$ since R is right Noetherian. This implies that $l_R(X_n) = l_R(X_{n+1}) = \ldots$, so

$$l_Q(X_n) = Q(l_Q(X_n) \cap R) = Ql_R(X_n) = Ql_R(X_{n+1}) = l_Q(X_{n+1}) = \ldots .$$

Thus Q satisfies the descending chain condition on left annihilators, so $Q/J(Q)$ is semisimple Artinian. This completes the proof.

PROPOSITION 2.11. *Let R be a ring such that $Z_l(R) = P(R)$, and let Q be the maximal left quotient ring of R. Q is a semiprimary right S-ring if and only if*

(1) *$QA = Q$ for every dense left ideal A of R,*
(2) *dense left ideals can be lifted modulo $P(R)$,*
(3) *$P(R)$ is nilpotent, and*
(4) *$R/P(R)$ is left Goldie.*

Proof. Suppose Q is a semiprimary right S-ring. Then $P(Q)$ is nilpotent, so $P(Q) \cap R$ is nilpotent. Therefore, $P(Q) \cap R \subseteq P(R)$. Also $P(R) = Z_l(R) = Z_l(Q) \cap R$, so $P(R) \subseteq QP(R) = Q(Z_l(Q) \cap R) = Z_l(Q)$. Since Q is semiprimary, $Z_l(Q)$ is a nil ideal, so $Z_l(Q) \subseteq J(Q) = P(Q)$. Hence

$$P(R) \subseteq Z_l(Q) \cap R \subseteq P(Q) \cap R,$$

so $P(R) = P(Q) \cap R$. Thus R satisfies conditions (1)–(3) by Theorem 2.2. Also

$$R/P(R) \cong (R + P(Q))/P(Q),$$

and $Q/P(Q)$ is a ring of left quotients of $(R + P(Q))/P(Q)$. Therefore, $R/P(R)$ is finite-dimensional since $Q/P(Q)$ is finite-dimensional. In addition, $Q/P(Q)$ satisfies the ascending chain condition on left annihilators, so $(R + P(Q))/P(Q)$ has the property since it is a subring of $Q/P(Q)$. Thus $R/P(R)$ is left Goldie, so R satisfies (4).

Assume conditions (1)–(4) are satisfied. By Theorem 2.2, it suffices to show that $P(R) = P(Q) \cap R$. Since $R/P(R)$ is a semiprime left Goldie ring and $((P(Q) \cap R) + P(R))/P(R)$ is a nil ideal in $R/P(R)$, we have

$$P(Q) \cap R \subseteq P(R).$$

Also $Z_l(Q) \cap R = Z_l(R) = P(R)$ is nilpotent, so $Z_l(Q)$ is nilpotent by the corollary to Lemma 1.15. Hence $Z_l(Q) \subseteq P(Q)$, so

$$P(R) = Z_l(Q) \cap R \subseteq P(Q) \cap R.$$

Consequently, $P(R) = P(Q) \cap R$. This completes the proof.

3. Quasi-Frobenius quotient rings.

Since the maximal left quotient ring Q of R is given by $Q = \{x \in E({}_RR) : (R : x)$ is a dense left ideal of $R\}$, $Q = E({}_RR)$ if and only if $(R : x)$ is a dense left ideal of R for every $x \in E({}_RR)$. Also $Q = E({}_RR)$ if and only if Q is left self-injective [**7**, Proposition 3, p. 95]. Using this fact, one can obtain the following two theorems. The proofs are contained in [**2**].

THEOREM 3.1. *Let Q be the maximal left quotient ring of R. Q is a left self-injective semiprimary ring if and only if*
(1) *$(R : x)$ is a dense left ideal of R for every $x \in E({}_RR)$,*
(2) *$QA = Q$ for every dense left ideal A of R, and*
(3) *$Z_l(R)$ is nilpotent.*

THEOREM 3.2. *Let Q be the maximal left quotient ring of R. Q is quasi-Frobenius if and only if*

(1) *$(R : x)$ is a dense left ideal of R for every $x \in E({}_RR)$,*
(2) *$QA = Q$ for every dense left ideal A of R,*
(3) *$Z_l(R)$ is nilpotent, and*
(4) *$R/[QZ_l(R)^k \cap R]$ is finite-dimensional* ($k = 1, 2, 3, \ldots$).

We will now turn to classical left quotient rings. An element $a \in R$ is regular if $l_R(a) = r_R(a) = (0)$. Q is the classical left quotient ring of R if $Q \supseteq R$, every regular element of R is invertible in Q, and given $q \in Q$, there exist $a, r \in R$, a regular, such that $q = a^{-1}r$. Lance Small [**15**; **16**] has given necessary and sufficient conditions for R to have a left Artinian classical left quotient ring Q.

THEOREM 3.3. *Suppose R has a left Artinian classical left quotient ring Q. Then:*
(1) *Q is a right S-ring if and only if $l_Rr_R(P(R)) = P(R)$.*
(2) *Q is a left S-ring if and only if $r_Rl_R(P(R)) = P(R)$.*

Proof. Let $P = P(Q)$ and $N = P(R)$. By the proof of Small's theorem, we know that $P \cap R = N$. Hence $P = QN$.

(1) Assume $l_Rr_R(N) = N$. Then $P = QN = Ql_Rr_R(N) = l_Qr_R(N) = l_Qr_R(QN) = l_Qr_R(P)$. But $r_R(P) \subseteq r_Q(P)$, so $l_Qr_R(P) \supseteq l_Qr_Q(P)$. Hence $l_Qr_Q(P) \subseteq l_Qr_R(P) = P \subseteq l_Qr_Q(P)$, so $P = l_Qr_Q(P)$. Also $P = J(Q)$ since Q is left Artinian. Therefore, Q is a right S-ring [**17**, Proposition 5.1].

Conversely, assume Q is a right S-ring. By [**17**, Proposition 5.1], P $= J(Q)$ is a left annihilator, say $P = l_Q(X)$. Since Q satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in X$ such that $P = l_Q(q_1) \cap \ldots \cap l_Q(q_n)$. There exist a, $y_i \in R$, a regular, such that $q_i = a^{-1}y_i$. Let $r \in l_R(y_1, \ldots, y_n)$. Then $raq_i = raa^{-1}y_i = ry_i = 0$ for each i, so $ra \in P$. Hence $r \in Pa^{-1} \cap R \subseteq P \cap R = N$. Thus $l_R(y_1, \ldots, y_n) \subseteq N$. Suppose $s \in N$. $sy_i = saa^{-1}y_i = saq_i$ for each i. Also $s \in N$ implies $sa \in N \subseteq P$, so $saq_i = 0$ for each i. Therefore, $sy_i = 0$ for each i, so $s \in l_R(y_1, \ldots, y_n)$. Thus $N \subseteq l_R(y_1, \ldots, y_n)$, so $N = l_R(y_1, \ldots, y_n)$. Therefore, $N = l_Rr_R(N)$.

(2) Suppose $r_Rl_R(N) = N$. We will first show that $l_Q(N) \subseteq l_Q(P)$. Since Q satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in P$ such that $l_Q(P) = l_Q(q_1, \ldots, q_n)$. Each q_i can be written as $q_i = a^{-1}x_i$, $a \in R$ regular, $x_i \in N$. Suppose $c^{-1}d \in l_Q(N)$. Then $(c^{-1}da)(a^{-1}x_i) = c^{-1}dx_i$ for each i. But $c^{-1}dN = (0)$ implies $dN = (0)$, so $dx_i = 0$ for each i. Hence $(c^{-1}da)q_i = 0$ for each i, so $c^{-1}da \in l_Q(P)$. Then $c^{-1}daP = (0)$ implies $daP = (0)$, so $daQP = (0)$ because $QP = P$. Also $aQ = Q$ since a is regular, so $(0) = dQP = dP$. Hence $d \in l_Q(P)$, so $c^{-1}d \in l_Q(P)$. Thus $l_Q(N) \subseteq l_Q(P)$. But $N \subseteq P$ implies $l_Q(N) \supseteq l_Q(P)$. Hence $l_Q(N) = l_Q(P)$. Now $N = r_Rl_R(N) = r_R(Ql_R(N)) = r_Rl_Q(N) = r_Rl_Q(P) = r_Ql_Q(P) \cap R$. Hence $P = QN = Q(r_Ql_Q(P) \cap R) = r_Ql_Q(P)$ since $r_Ql_Q(P)$ is a two-sided (therefore, left) ideal of Q. Then Q is a left S-ring by [**17**, Proposition 5.1].

Now suppose Q is a left S-ring. Then $r_Q l_Q(P) = P$ by [**17**, Proposition 5.1], so $N = r_Q l_Q(P) \cap R = r_R l_Q(P) = r_R[Ql_R(P)] = r_R l_R(P)$. Also $N \subseteq P$, so $l_R(N) \supseteq l_R(P)$. Hence $N = r_R l_R(P) \supseteq r_R l_R(N) \supseteq N$, so $N = r_R l_R(N)$. This completes the proof.

Suppose R has a classical left quotient ring, and let I be a left ideal of R. Then $I^T = \{r \in R : ar \in I$ for some regular element a of $R\}$. See [**8**] for details regarding the operator T. I is called a *closed left ideal* of R if $I^T = I$.

As an application of the preceeding theorem, we obtain the following result.

THEOREM 3.4. *Suppose R has a two-sided classical quotient ring Q which is left Artinian. Then Q is quasi-Frobenius if and only if R satisfies*

(*) *if $I \subseteq P(R)$ is a closed left (right) ideal of R, then I is a left (right) annihilator in R.*

Proof. First assume Q is quasi-Frobenius. Then every left (right) ideal of Q is a left (right) annihilator. Let $I \subseteq P(R)$ be a closed left ideal. Then $QI = l_Q(X)$ for some subset $X \subseteq Q$. Since Q satisfies the descending chain condition on left annihilators, there exist $q_1, \ldots, q_n \in X$ such that $QI = l_Q(q_1, \ldots, q_n)$. Since Q is a classical right quotient ring of R, there exist x_i, $b \in R$, b regular, such that $q_i = x_i b^{-1}$. Then $I = I^T = QI \cap R = l_R(x_1 b^{-1}, \ldots, x_n b^{-1}) = l_R(x_1, \ldots, x_n)$, so I is a left annihilator in R. Similarly, if $I \subseteq P(R)$ is a closed right ideal of R, then I is a right annihilator in R.

Now assume R satisfies (*). $P(R) = P(Q) \cap R$ since Q is left Artinian. Therefore, $P(R)$ is a closed left ideal and a closed right ideal, so $P(R)$ is a left and right annihilator by (*). Hence $l_R r_R(P(R)) = r_R l_R(P(R)) = P(R)$, so Q is a left and right S-ring by Theorem 3.3. $P(Q)$ is therefore a left annihilator and a right annihilator, so left and right annihilators can be lifted modulo $P(Q)$. Also $Q/P(Q)$ is semisimple Artinian. Therefore, if I is a left (right) ideal of Q such that $I \not\subseteq P(Q)$, then I is a left (right) annihilator in Q. Now let $I \subseteq P(Q)$ be a left ideal of Q. Then $I \cap R$ is a closed left ideal of R, and $I \cap R \subseteq P(R)$. By (*), $I \cap R = l_R(X)$ for some subset X of R. Then $I = l_Q(X)$, so I is a left annihilator in Q. Similarly, if $J \subseteq P(Q)$ is a right ideal in Q, then J is a right annihilator. Thus Q is a left Artinian ring such that every left ideal of Q is a left annihilator and every right ideal of Q is a right annihilator, so Q is quasi-Frobenius. This completes the proof.

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THE NUMBER OF GENERATORS OF A LINEAR p-GROUP

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Let G be a finite p-group, having a faithful character χ of degree f. The object of this paper is to bound the number, $d(G)$, of generators in a minimal generating set for G in terms of χ and in particular in terms of f. This problem was raised by D. M. Goldschmidt, and solved by him in the case that G has nilpotence class 2. (See [**1**, Lemma 2.8].) We obtain the following results:

THEOREM A. *Let χ be a faithful character of the p-group, G. Let $f = \chi(1)$ and let s be the number of linear constituents of χ. Then*
(a) $d(G) \leqq (3/p)(f - s) + s$. *Also,*
(b) if $p \geqq 3$ *and G is non-abelian, then* $d(G) \leqq f - p + 3$.

THEOREM B. *Let G be a p-group and let $\chi \in \mathrm{Irr}(G)$ be faithful. Then*

$$d(G) \leqq \frac{f + (f/p) + 2p - 4}{p - 1}.$$

It is shown by examples that the inequalities in Theorem A are best possible, and the one in Theorem B is nearly so.

1. Suppose χ is a faithful character of the p-group, G, and that $\chi = \psi + \lambda$, where λ is linear. Let $N = \mathrm{Ker}\,\psi$ so that λ_N is faithful and hence N is cyclic. It follows that $d(G) \leqq d(G/N) + 1$. By repeated application of this argument, we see that in order to prove Theorem A(a), it suffices to assume that χ has no linear constituents and show that $d(G) \leqq 3f/p$. Observe that part (b) of this theorem follows immediately from (a).

We would like to use reasoning similar to this in order to reduce the problem of bounding $d(G)$ to the situation of Theorem B, namely where χ is irreducible. In general, G is a subdirect product of the irreducible linear groups determined by the irreducible constituents of a faithful character. Unfortunately, if N_1, $N_2 \lhd G$ with $N_1 \cap N_2 = 1$, it does not follow that $d(G) \leqq d(G/N_1) + d(G/N_2)$. In order to overcome this difficulty we need to strengthen the theorem we are trying to prove.

Definition 1. Let G be a p-group and let $U \subseteq G$. Then

$$d_G(U) = d(U/(U \cap \Phi(G))).$$

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Instead of assuming that χ is faithful on G and bounding $d(G)$, we shall assume $U \lhd G$ and χ is a character of G with χ_U faithful and we shall bound $d_G(U)$. Since $d_G(G) = d(G)$, the new problem includes the old one.

LEMMA 2. *Let G be a p-group with $U \subseteq G$.*

(a) *If $U \subseteq H \subseteq G$, then $d_G(U) \leqq d_H(U)$ and $d_G(U) \leqq d_G(H)$.*

(b) *If $V \subseteq U$ and $V \lhd G$, then $d_G(U) = d_G(V) + d_{G/V}(U/V)$.*

Proof. (a). Since $H/H \cap \Phi(G)$ is elementary, $\Phi(H) \subseteq \Phi(G)$ and $U \cap \Phi(H) \subseteq U \cap \Phi(G)$. It follows that $d_H(U) \geqq d_G(U)$. Also, $d_G(U) = d(U\Phi(G)/\Phi(G)) \leqq d(H\Phi(G)/\Phi(G)) = d_G(H)$.

(b). Let $A = U \cap V\Phi(G)$. Then $U \supseteq A \supseteq U \cap \Phi(G)$ and $d_G(U) = d(U/A) + d(A/(U \cap \Phi(G)))$. Now $A = V(U \cap \Phi(G))$ and hence $A/(U \cap \Phi(G)) \cong V/(V \cap \Phi(G))$. Thus $d(A/(U \cap \Phi(G)) = d_G(V)$. Finally, we have $(U/V) \cap \Phi(G/V) = (U \cap V\Phi(G))/V = A/V$. Therefore, $d_{G/V}(U/V) = d((U/V)/(A/V)) = d(U/A)$. The proof is complete.

COROLLARY 3. *Let G be a p-group and let $U = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_n = 1$ where $N_i \lhd G$ for $1 \leqq i \leqq n$. Then*

$$d_G(U) = \sum_{i=1}^{n} d_{G/N_i}(N_{i-1}/N_i).$$

Proof. Repeated application of part (b) of the lemma yields the result.

Next, we wish to establish appropriate bounds when $\chi(1) = p$. The following lemma is well known and is stated here without proof.

LEMMA 4. *Let $A \lhd G$ be abelian with G/A cyclic. Let Ag be a generator of G/A. Then*

(a) $G' = \{a^{-1}a^g | a \in A\}$ *and*

(b) $|G'|\,|A \cap \mathbf{Z}(G)| = |A|$.

If χ is a character of a group, G, then det χ is the linear character of G obtained by taking the determinant of any representation of G which affords χ.

LEMMA 5. *Let G be a p-group with abelian $A \lhd G$ such that G/A is cyclic. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = p^e$ and suppose χ_A is faithful. Then*

(a) $d_G(A) \leqq e + 1$.

Also,

(b) *if* det $\chi_A = 1_A$, *then* $d_G(A) \leqq e$, *and*

(c) *if A has exponent $\leqq p^e$ then $d_G(A) \leqq e$.*

Proof. Let $Z = \mathbf{Z}(G) \cap A$. By Lemma 4, we have $|A : G'| = |Z|$. Since χ is irreducible, we have $Z(\mathrm{Ker}\ \chi)/\mathrm{Ker}\ \chi$ is cyclic and thus Z is cyclic since χ_A is faithful. If $|Z| \leqq p^e$, then $|A/(A \cap \Phi(G))| \leqq |A{:}G'| \leqq p^e$ and $d(A) \leqq e$. Therefore, (c) follows.

Now $\chi_Z = p^e\lambda$ where λ is a faithful character of Z. We have det $\chi_Z = \lambda^{p^e}$ and hence if det $\chi_Z = 1_Z$, it follows that $|Z| \leqq p^e$, and (b) now follows.

To prove (a), let C be the cyclic group of automorphisms of A induced by G/A. Since χ_A is faithful, C permutes the set of linear constituents of χ_A faithfully. This action is transitive, and hence regular and $|C| \leqq \chi(1)$. Let $\theta(a) = \Pi_{\sigma \in C} a^{\sigma}$ for $a \in A$. Then θ is an endomorphism of A and $\theta(a) = \theta(a^g)$ for $g \in G$. It follows that $G' \subseteq \operatorname{Ker} \theta = K$. It is clear that $\theta(A) \subseteq Z$ and since $|A : K| = |\theta(A)|$ and $|A : G'| = |Z|$, we have $|K : G'| = |Z : \theta(A)|$ and $A/K \cong \theta(A)$ is cyclic. If $Z = \langle z \rangle$, then $\theta(z) = z^{|C|}$ and hence $|Z : \theta(A)| \leqq |C| \leqq p^e$. It follows that $|K : K \cap \Phi(G)| \leqq p^e$ and $d_G(K) \leqq e$. Since $d_{G/K}(A/K) \leqq 1$, we have $d_G(A) \leqq e + 1$ and the proof is complete.

Lemma 6. *Let G be a p-group with $\chi \in \operatorname{Irr}(G)$ and $\chi(1) = p$. Let $U \lhd G$ and suppose χ_U is faithful. Then*

(a) *$d_G(U) \leqq 3$. Also,*

(b) *$d_G(U) \leqq 2$ if U is abelian, $\det \chi_U = 1_U$ or U has exponent p, and*

(c) *$d_G(U) \leqq 1$ if U is abelian and either $\det \chi_U = 1_U$ or U has exponent p.*

Proof. Use induction on $|G|$. If there exists $H \subset G$ with $U \subseteq H$ and χ_H irreducible, then the result follows since $d_G(U) \leqq d_H(U)$. Supposing, then, that $U \subset G$, we may assume that the restriction of χ to every maximal subgroup containing U is reducible. It follows that χ vanishes on $G - U\Phi(G)$ and hence $[\chi_{U\Phi(G)}, \chi_{U\Phi(G)}] = |G : U\Phi(G)|$. If $|G : U\Phi(G)| > p$, then $[\chi_U, \chi_U] = p^2$ and $\chi_U = p\lambda$, where λ is a faithful linear character of U. In this case U is cyclic and $d_G(U) \leqq 1$.

Under the assumption that $U \subset G$, the remaining case is where $|G : U\Phi(G)| = p$, G/U is cyclic, and U is abelian. In this case, Lemma 5 yields $d_G(U) \leqq 2$ and $d_G(U) \leqq 1$ if $\det \chi_U = 1_U$ or U has period p.

The only remaining case is where $U = G$. Here χ is faithful, and there exists an abelian subgroup A of index p (since χ is a monomial character). By the earlier cases, $d_G(A) \leqq 2$ and $d_G(A) \leqq 1$ if $\det \chi_A = 1_A$ or A has exponent p. The result now follows since $d_G(G) = d_G(A) + 1$.

2. In this section we prove Theorems A and B by working with irreducible characters, χ, of G which are faithful upon restriction to $U \lhd G$. In order to obtain the desired bound we introduce another parameter and prove a somewhat stronger theorem.

Theorem 7. *Let G be a p-group, $\chi \in \operatorname{Irr}(G)$ and $U \lhd G$ with χ_U faithful. Let $\chi(1) = f$ and let r be the number of (not necessarily distinct) irreducible constituents of χ_U. Set $b = (f + (f/p) + 2p - 4)/(p - 1)$. Then:*

(a) *$d_G(U) \leqq b$.*

(b) *If $r > 1$, then*

$$d_G(U) \leqq b - \frac{(r/p) - 1}{p - 1} - 1.$$

(c) *If $\det \chi_U = 1_U$, the inequalities in (a) and (b) may be replaced by strict inequalities.*

Proof. Use induction on $|U|\,|G|$. First note that if $f = 1$, then $b > 1$ and U is cyclic and the theorem holds. If $f = p$, then $b = 3$. In this case the theorem follows from Lemma 6. We therefore assume that $f \geqq p^2$.

If $r = 1$, then χ_U is irreducible and since $d_G(U) \leqq d_U(U)$, we are done by induction if $U < G$. Assume then, that $U = G$ and let H be a maximal subgroup of G, chosen so that χ_H is reducible. Since $|H|\,|G| < |G|\,|G|$, the inductive hypothesis applies and we conclude that $d_G(H) \leqq b - 1$ with strict inequality if det $\chi = 1_G$. It follows that $d_G(G) = 1 + d_G(H) \leqq b$, again with strict inequality if det $\chi = 1_G$. The theorem is now proved in this case.

Now suppose $r = p$. Choose a maximal subgroup, $H \supseteq U$. If χ_H is irreducible, we are done by applying the inductive hypothesis to H. We may assume, then, that $\chi_H = \theta_1 + \dots + \theta_p$, where the θ_i are conjugate irreducible characters of H. Since we are assuming $r = p$, we have $(\theta_i)_U$ irreducible for all i. On the other hand, since $f \geqq p^2$, $\theta_1(1) \geqq p$ and there exists a maximal subgroup, W, of H with $(\theta_1)_W$ reducible. It follows that $U \nsubseteq W$. Let λ be a linear character of H with kernel W and let $\psi = \lambda^G$ and $V = U \cap \operatorname{Ker}\psi$. Then $V \subseteq U \cap W \subset U$. Also, $\Phi(H) \subseteq W$ and $\Phi(H) \lhd G$, so that $\Phi(H) \subseteq \operatorname{Ker}\psi$ and consequently, U/V is elementary abelian. If ψ is reducible, then $W \lhd G$, $W = \operatorname{Ker}\psi$ and U/V is cyclic. If ψ is irreducible, there is a corresponding irreducible character $\hat{\psi}$ of G/V and $\hat{\psi}_{(U/V)}$ is faithful. It follows from Lemma 6(c) that $d_{G/V}(U/V) = 1$, and thus this is true in either case.

Since $V \subset U$, the theorem applies to bound $d_G(V)$. Since χ_V has at least p^2 irreducible constituents, we have $d_G(V) \leqq b - 2$, with strict inequality if det $\chi_U = 1_U$. Now $d_G(U) = d_G(V) + d_{G/V}(U/V) = 1 + d_G(V)$ and thus the theorem holds.

Finally, we assume that $r \geqq p^2$ and again choose a maximal $H \supseteq U$. As before, we may assume that $\chi_H = \theta_1 + \dots + \theta_p$. Let $\lambda_i = \det\theta_i$, let $\psi = \lambda_1{}^G$ and let $V = U \cap \operatorname{Ker}\psi$. If ψ is reducible then $\operatorname{Ker}\psi = \operatorname{Ker}\lambda_1$, U/V is cyclic and $d_{G/V}(U/V) = 1$. If ψ is irreducible, then as before we let $\hat{\psi}$ be the corresponding irreducible character of G/V. Since $\hat{\psi}_{(U/V)}$ is faithful and U/V is abelian, Lemma 6(b) yields $d_{G/V}(U/V) \leqq 2$. Now $\det\psi_H = \Pi\lambda_i = \det\chi_H$ and hence if det $\chi_U = 1_U$, it follows that $\det\hat{\psi}_{(U/V)} = 1_{(U/V)}$ and $d_{G/V}(U/V) \leqq 1$ by Lemma 6(c).

Now let $K_j = \operatorname{Ker}\theta_j$ and let $N_i = V \cap \bigcap_{j=1}^{i} K_j$. Set $N_0 = V$ and note that $N_p = 1$ since χ_V is faithful. By Corollary 3,

$$d_G(V) \leqq d_H(V) = \sum_{i=1}^{p} d_{H/N_i}(N_{i-1}/N_i).$$

Let r_i be the number of irreducible constituents of $(\theta_i)_{N_{i-1}}$ and observe that $r_i \geqq r/p \geqq p$. Let $\hat{\theta}_i$ be the irreducible character of H/N_i corresponding to θ_i for $1 \leqq i \leqq p$. We have $\hat{\theta}_{i(N_{i-1}/N_i)}$ is faithful and has trivial determinant since $N_{i-1} \subseteq V \subseteq \operatorname{Ker}\psi \subseteq \operatorname{Ker}\lambda_i$. It follows by the inductive hypothesis that

$$d_{H/N_i}(N_{i-1}/N_i) < \frac{(f/p) + (f/p^2) + 2p - 4}{p - 1} - \frac{(r_i/p) - 1}{p - 1} - 1.$$

Since $f \geqq p^2$ and $r_i/p \geqq r/p^2 \geqq 1$, the quantity on the right is an integer and we conclude

$$d_{H/N_i}(N_{i-1}/N_i) \leqq \frac{(f/p) + (f/p^2) + 2p - 4}{p - 1} - \frac{(r/p^2) - 1}{p - 1} - 2.$$

Therefore we have

$$\begin{aligned} d_G(V) &\leqq \frac{f + (f/p) + 2p^2 - 4p}{p - 1} - \frac{(r/p) - p}{p - 1} - 2p \\ &= \frac{f + (f/p) - p - (r/p)}{p - 1} = b - \frac{(r/p) - 1}{p - 1} - 3. \end{aligned}$$

Combining this inequality with $d_{G/V}(U/V) \leqq 2$ and $d_{G/V}(U/V) \leqq 1$ if $\det \chi_U = 1_U$, yields (b) and (c) in this case. The proof of the theorem is now complete.

Observe that Theorem B is a special case of Theorem 7(a) and has therefore now been proved. Also note that if $f \geqq p$, we have

$$\frac{f + (f/p) + 2p - 4}{p - 1} \leqq \frac{3f}{p}.$$

Proof of Theorem A. It has already been noted that it suffices to prove (a), and that, only when χ has no linear constituents. Let $\chi_1, \chi_2, \ldots, \chi_n$ be the distinct irreducible constituents of χ and let $K_j = \operatorname{Ker} \chi_j$ and $N_i = \bigcap_{j=1}^{i} K_j$. Then by Corollary 3, $d(G) = \Sigma\, d_{G/N_i}(N_{i-1}/N_i)$ where $N_0 = G$. By Theorem 7 applied to G/N_i, we have $d_{G/N_i}(N_{i-1}/N_i) \leqq 3\chi_i(1)/p$. It follows that $d(G) \leqq 3\chi(1)/p$ as desired.

We end this section with a corollary of Theorem 7. The bound given here will be shown to be sharp.

COROLLARY 8. *Let G be a p-group and let $U \lhd G$ be abelian. Suppose $\chi \in \operatorname{Irr}(G)$ with $\chi(1) = f$ and χ_U faithful. Then $d_G(U) \leqq (f - 1)/(p - 1) + 1$.*

Proof. If $f = 1$, U is cyclic. Otherwise, apply Theorem 7(b) with $r = f$.

3. In this section we discuss some examples.

THEOREM 9. *The bounds given in Theorem* A *are sharp.*

Proof. Let H be the central product of a non-abelian group of order p^3 with a cyclic group of order p^2. Then $d(H) = 3$ and H has a faithful irreducible character of degree p. Now let G be the direct product of $(f - s)/p$ copies of H and s copies of a cyclic group of order p. Then $d(G) = 3(f - s)/p + s$ and G has a faithful character of degree f.

The direct product of one copy of H with $f - p$ cyclic groups of order p shows that the bound in (b) is the best possible.

THEOREM 10. *The bound given in Lemma* 5(a) *is sharp.*

Proof. We need an example of a p-group G with $A \lhd G$, A abelian, G/A cyclic, $\chi \in \operatorname{Irr}(G)$, χ_A faithful, $\chi(1) = p^e$ and $d_G(A) = e + 1$. The example is as follows.

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Let $A = \langle x_1 \rangle \times \langle x_2 \rangle \times \ldots \times \langle x_{e+1} \rangle$, where the order, $o(x_i) = p^i$. Define an automorphism, σ, of A by

$$x_i^\sigma = x_i x_{i+1}^p \text{ for } 1 \leqq i \leqq e$$

and $x_{e+1}^\sigma = x_{e+1}$. We claim that $o(\sigma) \leqq p^e$. Let $Z = \langle x_{e+1} \rangle$. Then σ acts on A/Z and this is the situation corresponding to the case $e - 1$. By induction, then, $\sigma^{p^{e-1}}$ acts trivially on A/Z. Let $\theta = \sigma^{p^{e-1}}$ so that $a^{-1}a^\theta \in Z$ for all $a \in A$.

Now let $\bar{A} = A/\Omega_1(A)$. Then σ acts on $\bar{A}$ and this too is the situation corresponding to $e - 1$. Thus θ is trivial on $\bar{A}$ and $a^{-1}a^\theta \in \Omega_1(A) \cap Z$ for all $a \in A$. If $a^\theta = ay$, then $y^p = 1$ and $y^\theta = y$ so that $a^{\theta p} = ay^p = a$, and $o(\sigma) \leqq p^e$ as claimed.

Let G be the semi-direct product, $A \times\!| \langle \sigma \rangle$. It is clear that $G' = \Phi(A)$ and hence $|A : G'| = p^{e+1}$. By Lemma 4, $|A \cap \mathbf{Z}(G)| = p^{e+1}$. However, since $\langle \sigma \rangle$ acts faithfully on A, we have $\mathbf{Z}(G) \subseteq A$. Since $Z \subseteq \mathbf{Z}(G)$ and $|Z| = p^{e+1}$, it follows that $\mathbf{Z}(G) = Z$ is cyclic. Therefore, G has a faithful irreducible character χ with $\chi(1) \leqq |G : A| \leqq p^e$. Finally, since $G' = \Phi(A)$, it follows that $d_G(A) = d(A) = e + 1$. By Lemma 5(a), $\chi(1) = p^e$ and the proof is complete.

THEOREM 11. *Let E be an elementary abelian p-group of order p^k, $k \geqq 1$. There exists an abelian p-group, U, on which E acts so that*

(a) *$\mathbf{C}_U(E)$ is cyclic*

and

(b) *$d(U/[U, E]) = (p^k - 1)/(p - 1) + 1$.*

Before proving Theorem 11, we discuss some consequences. Let G be the semi-direct product $U \times\!| E$. Then we have $G' = [U, E]$ and $G/G' \cong U/[U, E] \times E$. It follows that $d_G(U) = d(U/[U, E]) = (p^k - 1)/(p - 1) + 1$ and that $d(G) = d_G(U) + k$. Now $\mathbf{Z}(G) \cap U = \mathbf{C}_U(E)$ is cyclic, and thus there exists $\chi \in \mathrm{Irr}(G)$ with $\mathbf{C}_U(E) \cap \operatorname{Ker} \chi = 1$. It follows that χ_U is faithful. Let $f = \chi(1)$ so that $f \leqq |G : U| = p^k$. On the other hand, Corollary 8 asserts that $d_G(U) \leqq (f - 1)/(p - 1) + 1$. It follows that $f = p^k$. At this point we have proved

COROLLARY 12. *The bound of Corollary 8 is sharp.*

In the above situation, $f = |G : U|$ and it follows that U is a maximal abelian subgroup of G. Therefore, $\mathbf{C}_U(E) = \mathbf{Z}(G)$ and hence χ is faithful. Let $b = b(f)$ be the bound given in Theorem B. If $f = p$ or p^2, we see that $d(G) = b$. Although the above group, G, does not prove that the bound, b, is sharp; it does show that it is not far wrong, since for $f > 1$ we have $d(G) > pb/(p + 1)$.

Before proving Theorem 11, we need the following counting lemma.

LEMMA 13. *Let n and k be positive integers and let N be the number of k-tuples, $(x_1, \ldots, x_k)$ of integers, $0 \leqq x_i \leqq n$, such that $\Sigma x_i \equiv 0 \bmod n$. Then*

$$N = \frac{(n + 1)^k - 1}{n} + 1.$$

Proof. We count the k-tuples with $\Sigma\, x_i \equiv 0 \bmod n$ according to the number, r, of entries equal to n. If $r = k$, there is one such k-tuple. If $r < k$, the number of k-tuples with the required property is $\binom{k}{r}F(r)$ where $F(r)$ is the number of $(k - r)$-tuples, $(y_1, \dots, y_{k-r})$, where $0 \leqq y_i \leqq n - 1$ and $\Sigma\, y_i \equiv 0 \bmod n$.

We may identify the n^{k-r} $(k - r)$-tuples of integers y_i, $0 \leqq y_i \leqq n - 1$ with the elements of the direct product of $k - r$ cyclic groups of order n. Under this identification, the tuples, $(y_1, \dots, y_{k-r})$, with $\Sigma\, y_i \equiv 0 \bmod n$, correspond to the elements of the kernel of a homomorphism onto the cyclic group of order n. It follows that $F(r) = n^{k-r-1}$ and

$$N = 1 + \sum_{r=0}^{k-1} \binom{k}{r} n^{k-r-1}$$

$$= 1 + \frac{1}{n}\left((n + 1)^k - 1\right),$$

as desired.

Proof of Theorem 11. We shall construct U as an (additive) subgroup of the group ring $R[E] = A$, where $R = \mathbf{Z}/p^{k+1}\mathbf{Z}$. Now E acts on A by right multiplication and $\mathbf{C}_A(E) = R(\sum_{x \in E} x)$, a cyclic group. Therefore, it suffices to find a subgroup $U \subseteq A$ which is invariant under E (i.e., U must be an ideal) such that $d(U/[U, E]) = (p^k - 1)/(p - 1) + 1$.

First we observe that for $x \in E$, we have $(x - 1)^p = p\sum_{i=1}^{p-1} r_i(x - 1)^i$ for suitable $r_i \in R$. This is so because of the polynomial identity $X^p - (X + 1)^p + 1 = p \sum_{i=1}^{p-1} m_i X^i$ where $m_i = -\binom{p}{i}/p \in \mathbf{Z}$. Substituting $x - 1$ for X yields the required result.

Next we establish some notation. Let $\{x_1, \dots, x_k\}$ be a fixed set of generators for E. Let $\mathscr{S} = \{(m_1, \dots, m_k) | m_i \in \mathbf{Z},\ 0 \leqq m_i \leqq p - 1\}$. If $s = (m_1, \dots, m_k) \in \mathscr{S}$, we write $\sum s$ for $\sum m_i$ and $(x - 1)^s$ for $(x_1 - 1)^{m_1}(x_2 - 1)^{m_2} \dots (x_k - 1)^{m_k} \in A$.

We claim that $\{(x - 1)^s | s \in \mathscr{S}\}$ is an R-basis for A. Since $|\mathscr{S}| = p^k = |E|$, it suffices to show that if $\sum_{s \in \mathscr{S}} r_s(x - 1)^s = 0$ with $r_s \in R$, then all $r_s = 0$. Suppose, then, that some $r_s \neq 0$. By multiplying the dependence by the highest power of p which fails to annihilate all of the coefficients, we may assume that $pr_s = 0$ for all $s \in \mathscr{S}$. Now, among all $s \in \mathscr{S}$ with $r_s \neq 0$, choose one, say $s_0 = (m_1, \dots, m_k)$, with $\sum s_0$ minimal. Let $t = (p - 1 - m_1, \dots, p - 1 - m_k) \in \mathscr{S}$ and multiply the dependence by $(x - 1)^t$. Observe that $r_s(x - 1)^s (x - 1)^t = 0$ if $s \neq s_0$. This is so because if $s \neq s_0$ and $r_s \neq 0$, then $\sum s \geqq \sum s_0$ and hence some entry (say the ith) in the k-tuple, s, is strictly larger than the corresponding entry in s_0. It follows that $(x - 1)^s(x - 1)^t \in (x_i - 1)^p A \subseteq pA$. Since $pr_s = 0$, it follows that $r_s(x - 1)^s(x - 1)^t = 0$. We now have

$$0 = r_{s_0}(x - 1)^{s_0}(x - 1)^t = r_{s_0}(x - 1)^{p-1} \dots (x_k - 1)^{p-1}.$$

This is a contradiction, since 1 is clearly in the support of $(x_1 - 1)^{p-1} \dots (x_k - 1)^{p-1}$ and $r_{s_0} \neq 0$.

We now use this basis for A to construct two subgroups. For $s \in \mathscr{S}$, let $l(s) = l$ be the unique integer such that $l(p-1) \leqq \sum s < (l+1)(p-1)$ and let $m(s) = m$ be the unique integer such that $m(p-1) < \sum s \leqq (m+1)(p-1)$. Note that $0 \leqq l(s) \leqq k$ and $-1 \leqq m(s) \leqq k-1$. Also $l(s) = m(s)$ unless $\sum s$ is a multiple of $(p-1)$, in which case $m(s) = l(s) - 1$. Now set

$$U = \{p^{k-l(s)}(x-1)^s \mid s \in \mathscr{S}\}$$

and

$$V = \{p^{k-m(s)}(x-1)^s \mid s \in \mathscr{S}\}.$$

It is clear that U is the direct sum of the cyclic groups generated by the given set of generators of U and V is the sum of the subgroups of these cyclic groups generated by the generators of V. It follows that $d(U/V)$ is equal to the number of the generators of U which do not lie in V. This is exactly the number of $s \in \mathscr{S}$ with $\sum s \equiv 0 \mod p-1$. By Lemma 13, we have $d(U/V) = (p^k - 1)/(p-1) + 1$.

The proof will be complete when we show $[U, E] = V$ because it then follows automatically that U is E-invariant. Now if $s, s' \in \mathscr{S}$ with $\sum s = 1 + \sum s'$, then $m(s) = l(s')$. If $s \neq (0, 0, \ldots, 0)$, we can choose i, and $s' \in \mathscr{S}$ with $(x-1)^s = (x-1)^{s'}(x_i - 1)$ and $\sum s = 1 + \sum s'$. Thus $p^{k-m(s)}(x-1)^s = p^{k-l(s')}(x-1)^{s'}(x_i-1)$. It follows that every generator of V is of the form $u(x_i - 1)$ for some generator u of U. (If $s = (0, 0, \ldots, 0)$, then $p^{k-m(s)}(x-1)^s = 0$.) Therefore, $V \subseteq [U, E]$. The generators u which arise this way are exactly those which correspond to $s' \in \mathscr{S}$ where the ith entry of s' is $< p-1$. For each such u, we therefore have $u(x_i - 1) \in V$.

All that remains now in order to prove that $[U, E] \subseteq V$ is to show that $p^{k-l(s)}(x-1)^s(x_i - 1) \in V$ whenever the ith entry of s is equal to $p-1$. Recall that

$$(x_i - 1)^p = p \sum_{j=1}^{p-1} r_j(x_i - 1)^j,$$

and thus it follows that

$$(x-1)^s(x_i - 1) = p \sum_{j=1}^{p-1} r_j(x-1)^{s_j}$$

where $s_j \in \mathscr{S}$ and $\sum s_j = j + \sum s - (p-1) > \sum s - (p-1)$. Therefore $m(s_j) \geqq l(s) - 1$ and

$$p^{k-l(s)}(x-1)^s(x_i-1) = \sum_{j=1}^{p-1} r_j p^{k-l(s)+1}(x-1)^{s_j} \in V.$$

The proof of the theorem is now complete.

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ANALYTIC TOEPLITZ AND COMPOSITION OPERATORS

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1. Introduction. This paper is a continuation of [**1**] where we began the study of intertwining analytic Toeplitz operators. Recall that X *intertwines* two operators A and B if $XA = BX$. Let H^2 be the Hilbert space of analytic functions in the open unit disk D for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in the L^2 norm, and H^∞ be the set of bounded functions in H^2. For $\varphi \in H^\infty$, T_φ(or $T_{\varphi(z)}$) is the *analytic Toeplitz operator* defined on H^2 by the relation $(T_\varphi f)(z) = \varphi(z)f(z)$. For $\varphi \in H^\infty$, we shall denote $\{\varphi(z): |z| < 1\}$ by Range (φ) or $\varphi(D)$. Then $\sigma_p(T_\varphi^*) \supseteq \tilde{\varphi}(D)$ where $\tilde{\varphi}(z) = \overline{\varphi(\bar{z})}$ and $\sigma(T_\varphi) =$ Closure$(\varphi(D))$ [**1**]. If $\varphi \in H^\infty$ maps D into D, then we define the *composition operator* C_φ on H^2 by the relation $(C_\varphi f)(z) = f(\varphi(z))$. J. Ryff has shown [**11**, Theorem 1] that C_φ is a bounded linear operator on H^2. In § 2 we investigate intertwining operators between analytic Toeplitz operators using composition operators, and in § 3 we study a special class of composition operators.

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2. Intertwining analytic Toeplitz operators.

THEOREM 1 (see [**1**]). *Let $\varphi, \psi \in H^\infty$. If $\psi(D) \not\subseteq \sigma(T_\varphi)$, then the only bounded linear operator X satisfying $XT_\varphi = T_\psi X$* is $X = 0$.

COROLLARY 1. *If $\varphi, \psi \in H^\infty$ are such that there exists $X \neq 0$, $Y \neq 0$ satisfying $XT_\varphi = T_\psi X$ and $T_\varphi Y = YT_\psi$, then $\sigma(T_\varphi) = \sigma(T_\psi)$.*

Proof. Applying Theorem 1 we see that $\psi(D) \subseteq \sigma(T_\varphi)$ and $\varphi(D) \subseteq \sigma(T_\psi)$. Since $\sigma(T_\varphi) =$ Closure$(\varphi(D))$, $\sigma(T_\varphi) = \sigma(T_\psi)$.

PROPOSITION 1. *Let $\varphi, \psi \in H^\infty$. If there exists an analytic function ω mapping D into D such that $\varphi(\omega(z)) = \psi(z)$, then there exists a nonzero X such that $XT_\varphi = T_\psi X$.*

Proof. Since C_ω is clearly nonzero and since for $f \in H^2$

$$((C_\omega T_\varphi)f)(z) = \varphi(\omega(z))f(\omega(z)) = \psi(z)f(\omega(z)) = (T_\psi C_\omega f)(z),$$

we have that

$$C_\omega T_\varphi = T_\psi C_\omega.$$

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THEOREM 2. *Let $\varphi, \psi \in H^\infty$, φ univalent in D. Then $\psi(D) \not\subseteq \varphi(D)$ if and only if $XT_\varphi = T_\psi X$ implies $X = 0$. In addition, $\tilde{\varphi}(D) = \sigma_p(T_\varphi^*)$.*

Proof. Suppose $\psi(D) \not\subseteq \varphi(D)$.

Case 1. ψ is constant, $\psi(z) = \lambda$. Then either $\lambda \notin \sigma(T_\varphi)$ in which case $X = 0$ by Theorem 1, or $\lambda \in \sigma(T_\varphi)\backslash\varphi(D)$. Suppose X satisfies $XT_\varphi = T_\psi X = \lambda X$. Then $(T_\varphi^* - \lambda^*)X^* = 0$, so that Range $X^* \subseteq \text{Null}(R_\varphi^* - \lambda^*) =$ Range $(T_\varphi - \lambda)^\perp$. Since $\lambda \notin \varphi(D)$, the univalent function $\varphi - \lambda$ never vanishes in D. Hence $\varphi - \lambda$ contains no Blaschke products, and by Theorem 3.17 in [**4**] (see also [**9**]) $\varphi - \lambda$ contains no singular inner factor. Thus the decomposition of H^2 functions into the product of an inner and an outer function [**7**, p. 67] implies that $\varphi - \lambda$ must be outer. But if $\varphi - \lambda$ is outer, then Range $(T_\varphi - \lambda)$ is dense in H^2 [**7**, p. 101], so that Range $X^* = \{0\}$. Thus $X = 0$. This also establishes that $\sigma_p(T_\varphi^*) = \tilde{\varphi}(D)$.

Case 2. ψ is not constant. Now $N = \psi(D) \cap \mathbf{C}\backslash\varphi(D)$ is nonempty by hypothesis. Since φ is a univalent analytic function, $\varphi(D)$ is an open simply connected set, hence $\mathbf{C}\backslash\varphi(D)$ contains no isolated points. Since ψ is nonconstant, $\psi(D)$ is an open set. Thus N is the nonempty intersection of an open set and a closed set containing no isolated points, and hence N must be uncountable. The proof of Theorem 1 then implies $X = 0$.

Suppose $\psi(D) \subseteq \varphi(D)$. Since φ is univalent, $F(z) = \varphi^{-1}(\psi(z))$ is an analytic function mapping D into D such that $\varphi(F(z)) = \psi(z)$. Hence Proposition 1 implies there exists an $X \neq 0$ such that $XT_\varphi = T_\psi X$.

PROPOSITION 2. *Let $\varphi, \psi \in H^\infty$ map D into D. If $\overline{C_\varphi H^2}$ reduces T_φ and if there exists $K > 0$ such that*

$$(*) \qquad \| C_\psi g \| \leqq K \| C_\varphi g \| \text{ for all } g \in H^2,$$

then there exists a bounded $X \neq 0$ such that $XT_\varphi = T_\psi X$. (We remark that $(*)$ is equivalent to the existence of $Y \in \mathscr{B}(H)$ satisfying $YC_\varphi = C_\psi$ and to $C_\psi^* H^2 \subseteq C_\varphi^* H^2$ [**2**].)

Proof. Write $H^2 = \overline{C_\varphi H^2} \oplus (C_\varphi H^2)^\perp$ and define X on $C_\varphi H^2 \oplus (C_\varphi H^2)^\perp$ by

$$\begin{aligned} X(C_\varphi g) &= C_\psi g \quad \text{for } g \in H^2 \\ Xf &= 0 \qquad \text{for } f \perp C_\varphi H^2. \end{aligned}$$

Then X is well defined and $(*)$ implies that X is bounded, so we can continuously extend it to all of H^2. Also

$$\begin{aligned} (XT_\varphi)f &= XT_\varphi(C_\varphi g \oplus h) = X(\varphi C_\varphi g \oplus \varphi h) \\ &= X(\varphi C_\varphi g) = \psi C_\psi g \end{aligned}$$

and

$$\begin{aligned} (T_\psi X)f &= T_\psi X(C_\varphi g \oplus h) = T_\psi X C_\varphi g \\ &= T_\psi C_\psi g = \psi C_\psi g. \end{aligned}$$

Hence $XT_\varphi = T_\psi X$ on $C_\varphi H^2 \oplus (C_\varphi H^2)^\perp$ and thus on H^2.

Remarks. 1. There is no loss of generality in assuming φ, ψ map D into D, since $\tilde{\varphi} = \varphi/2M$ and $\tilde{\psi} = \psi/2M$, where $M = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}$, map D into D, and $XT_{\tilde{\varphi}} = T_{\tilde{\psi}}X$ if and only if $XT_\varphi = T_\psi X$.

2. $\overline{C_\varphi H^2}$ is always invariant for T_φ, since $T_\varphi C_\varphi = C_\varphi T_z$. However, $\overline{C_\varphi H^2}$ need not always reduce T_φ (example: if $\varphi(z) = \frac{1}{2}z^2 + \frac{1}{2}z^3$ then $e_1(z) = z \in \text{Null}\,(C_\varphi{}^*) = (C_\varphi H^2)^\perp$ but $C_\varphi{}^* T_\varphi e_1 = \frac{1}{2}e_1 \neq 0$).

3. Nevertheless there are examples where $\overline{C_\varphi H^2}$ reduces T_φ. If $C_\varphi H^2$ is dense, then $\overline{C_\varphi H^2}$ trivially reduces T_φ. If φ is an inner function, then $\overline{C_\varphi H^2}$ reduces T_φ since, in this case, $T_\varphi{}^* C_\varphi = C_\varphi(T_z{}^* + \bar{\varphi}(0)E)$ where $(Ef)(z) = f(0)$. Also, if ω is an inner function and $C_\psi H^2$ is dense in H^2, then $\overline{C_\varphi H^2}$ reduces T_φ for $\varphi(z) = \psi(\omega(z))$.

COROLLARY 2. *Let* φ, $\psi \in H^\infty$, φ *an inner function. Then* $\tilde{\psi}(D) \not\subseteq \sigma_p(T_\varphi{}^*)$ *if and only if* $XT_\varphi = T_\psi X$ *implies* $X = 0$.

Proof. If φ is constant the statement is clear, so we assume φ is nonconstant. Hence $\sigma_p(T_\varphi{}^*) = D$[**5**, p. 230].

Suppose $\tilde{\psi}(D) \subseteq D$. By Remark 3, $\overline{C_\varphi H^2}$ reduces T_φ, and by Theorem 1 in [**10**], C_φ is bounded below, hence Proposition 2 implies there exists $X \neq 0$ such that $XT_\varphi = T_\psi X$. An alternative proof is to observe that there exists $Y \neq 0$ such that $YT_\varphi = T_z Y$, since T_φ and T_z are both isometries. Hence $X = C_\psi Y \neq 0$ satisfies $XT_\varphi = T_\psi X$.

Suppose $\tilde{\psi}(D) \not\subseteq D$. The result then follows from Corollary 1 in [**1**] with (i) replaced by

$$\text{(i)}' \qquad \text{Interior }(\text{Closure}(\sigma_p(T_\varphi{}^*))) = \sigma_p(T_\varphi{}^*).$$

In [**1**] we conjectured that $\tilde{\psi}(D) \not\subseteq \sigma_p(T_\varphi{}^*)$ is necessary and sufficient for $XT_\varphi = T_\psi X$ to imply $X = 0$. Theorem 2 and Corollary 2 establish this conjecture if φ is univalent or inner. In case φ is a polynomial, $\tilde{\varphi}(D) = \sigma_p(T_\varphi{}^*)$ (see [**3**]). Since it can be shown that Interior $(\text{Closure}(\varphi(D))) = \varphi(D)$, Corollary 1 in [**1**] implies the sufficiency of our conjecture in case φ is a polynomial.

3. Composition operators.

In this section we study the special class of composition operators C_φ of the form $\varphi(z) = \alpha + \beta z$, that is, $|\alpha| < 1$, $|\alpha| + |\beta| \leq 1$. E. Nordgren [**10**] has studied C_φ when φ is an inner function, while H. Schwartz [**12**] has obtained numerous results concerning composition operators.

THEOREM 3. (i) *If* $|\beta| = 1$, *then* $C_{\alpha+\beta z}$ *is a unitary operator whose spectrum is the closure of the set* $\{1, \beta, \beta^2, \ldots\}$.

(ii) *If* $|\alpha| + |\beta| < 1$, *then* $C_{\alpha+\beta z}$ *is a compact operator whose spectrum is the closure of* $\{1, \beta, \beta^2, \ldots\}$.

(iii) *If* $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, β *not positive, then* $C_{\alpha+\beta z}$ *is a noncompact operator, whose square is compact, and whose spectrum is the closure of* $\{1, \beta, \beta^2, \ldots\}$.

(iv) *If* $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, β *positive, then* $C_{\alpha+\beta z}$ *is a cosubnormal operator whose spectrum is the closed disk of radius* $\beta^{-\frac{1}{2}}$ *centered at the origin.*

Proof. Before beginning the proof, notice that under the natural identification between H^2 and $l_+{}^2$ (i.e., $\sum_{n=0}^{\infty} a_n z^n \rightarrow \{a_n\}_0^{\infty}$), $C_{\alpha+\beta z}$ has a matrix representation on $l_+{}^2$ as

$$C_{\alpha+\beta z} \cdot \sim \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \ldots \\ & \beta & 2\alpha\beta & 3\alpha^2\beta \ldots \\ & & \beta^2 & 3\alpha\beta^2 \ldots \\ \text{\Large 0} & & & \beta^3 \ddots \end{bmatrix}$$

that is, $C_{\alpha+\beta z} \sim (a_{ij})$ where $a_{ij} = 0$ if $j < i$ and $a_{ij} = \binom{j}{i}\alpha^{j-i}\beta^i$ if $j \geqq i$.

Proof of 3(i). Since $|\beta| = 1$, α equals 0. Hence $C_{\alpha+\beta z}$ corresponds to a diagonal matrix all of whose entries have modulus 1. Thus $C_{\alpha+\beta z}$ is unitary with spectrum = Closure(Diagonal) = Closure$(1, \beta, \beta^2, \ldots)$.

Proof of 3(ii). Since $|\alpha| + |\beta| = r < 1$, we have $|\alpha + \beta z| \leqq r < 1$ for $|z| \leqq 1$. Hence Theorem 5.2 in [**12**] implies that $C_{\alpha+\beta z}$ is compact with spectrum = Closure$\{1, \beta, \beta^2, \ldots\}$, and that if $\beta \neq 0$ then each β^n is a simple eigenvalue. An alternative proof is to first notice that $\sigma_p(C_{\alpha+\beta z}) \supseteq \{1, \beta, \beta^2, \ldots\}$. In fact, if $f_n(z) = (z - \alpha/(1-\beta))^n$ then $C_{\alpha+\beta z} f_n = \beta^n f_n$. Next notice that the matrix (a_{ij}) of $C_{\alpha+\beta z}$ satisfies $\sum_{i,j=0}^{\infty} |a_{ij}| = 1/(1-r) < \infty$, so that $C_{\alpha+\beta z}$ is compact. From this it is not hard to conclude that spectrum = Closure $\{1, \beta, \beta^2, \ldots\}$ and that each eigenvalue is simple if $\beta \neq 0$.

Proof of 3(iii). Since $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, and β is not positive, we have $|1 + \beta| < 1 + |\beta|$ and hence $|\alpha(1+\beta)| + |\beta^2| < 1$. Because $C_{\alpha+\beta z}{}^2 = C_{\alpha(1+\beta)+\beta^2 z}$, 3(ii) and the spectral mapping theorem [**5**, p. 38] imply that $C_{\alpha+\beta z}{}^2$ is compact and that

$$(\sigma(C_{\alpha+\beta z}))^2 = \sigma(C_{\alpha+\beta z}{}^2) = \sigma(C_{\alpha(1+\beta)+\beta^2 z}) = \text{Closure}\{1, \beta^2, \beta^4, \ldots\}.$$

Hence

$$\sigma(C_{\alpha+\beta z}) \subseteq \text{Closure}\{\pm 1, \pm\beta, \pm\beta^2, \ldots\}.$$

As usual, $\sigma_p(C_{\alpha+\beta z}) \supseteq \{1, \beta, \beta^2, \ldots\}$. Recall that β^{2n} is a simple eigenvalue for $C_{\alpha+\beta z}{}^2$ with eigenvector $f_n(z) \equiv (z - \alpha(1+\beta)/(1-\beta^2))^n = (z - \alpha/(1-\beta))^n$, which is also the eigenvector for $C_{\alpha+\beta z}$ corresponding to the eigenvalue β^n. Hence

$$\mathcal{N} \equiv \text{Null}(C_{\alpha+\beta z}{}^2 - \beta^{2n}) = \text{Null}(C_{\alpha+\beta z} - \beta^n),$$

and

$$\text{Null}(C_{\alpha+\beta z} + \beta^n) = \{0\},$$

since $\text{Null}(C_{\alpha+\beta z} + \beta^n) \subseteq \text{Null}(C_{\alpha+\beta z}{}^2 - \beta^{2n})$ and $\beta \neq 0$. We need to show that $-\beta^n \notin \sigma(C_{\alpha+\beta z})$ for $n = 0, 1, 2, \ldots$. If $-\beta^n \in \sigma(C_{\alpha+\beta z})$, then $-\beta^n \in \partial\sigma(C_{\alpha+\beta z}) \subset \sigma_a(C_{\alpha+\beta z})$ [**5**, p. 39]. Hence there exist y_m, $\|y_m\| = 1$ such that

$$\|(C_{\alpha+\beta z} + \beta^n) y_m\| \rightarrow 0 \text{ as } m \rightarrow \infty$$

so

$$\|(C_{\alpha+\beta z}{}^2 - \beta^{2n}) y_m\| \rightarrow 0 \text{ as } m \rightarrow \infty.$$

Let $y_m = y'_m \oplus y''_m \in \mathcal{N} \oplus \mathcal{N}^{\perp}$. Since $C_{\alpha+\beta z}{}^2$ is compact and $\beta \neq 0$, $C_{\alpha+\beta z}{}^2 - \beta^{2n}$ is bounded below on $\mathcal{N}^{\perp}$ [**5**, p. 91]. Hence $y_m'' \to 0$. Because $1 = \|y_m\|^2 + \|y_m\|^2$, there is a subsequence $\{y'_{mk}\}$ that converges weakly to g_n where $g_n \in \mathcal{N}$, $\|g_n\| = 1$. Hence

$$(C_{\alpha+\beta z} + \beta^n)y_{mk} \to (C_{\alpha+\beta z} + \beta^n)g_n = 0,$$

which contradicts $\mathrm{Null}(C_{\alpha+\beta z} + \beta^n) = \{0\}$. Thus $\sigma(C_{\alpha+\beta z}) = \mathrm{Closure}\{1, \beta, \beta^2, \ldots\}$.

In order to see that $C_{\alpha+\beta z}$ is not compact, we employ the argument on page 23 of [**12**]. By hypothesis $|\alpha| + |\beta| = 1$, $|\beta| \neq 1$, so that $\alpha = \rho e^{i\theta}$ and $\beta = (1-\rho)e^{i\eta}$ where $0 < \rho < 1$. If we define $f_n(z) = 1/\sqrt{n}\ (e^{i\theta} - z + z/n)^{-1}$ then $f_n \in H^2$, $\frac{1}{2} \leqq \|f_n\|^2 \leqq 1$, and $f_n \to 0$ uniformly on compact subsets of D. Also $\|C_{\alpha+\beta z} f_n\|^2 \geqq \|f_n\|^2 \geqq \frac{1}{2}$. Theorem 2.5 in [**12**] then implies that $C_{\alpha+\beta z}$ is not compact.

Proof of 3(iv). We first consider the case when α is positive. Then $\alpha + \beta = 1$. Define $C_0{}^*$ to be that operator on H^2 whose matrix representation under the natural identification between H^2 and $l_+{}^2$ is

$$C_0{}^* \sim \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ & \frac{1}{2} & \frac{1}{3} & \cdots \\ & 0 & \frac{1}{3} & \cdots \\ & & & \ddots \end{bmatrix}$$

That is, $C_0{}^* \sim (b_{ij})$ where $b_{ij} = 0$ if $j < i$ and $b_{ij} = 1/j$ if $j \geqq i$. Then $C_0{}^*$ is a bounded linear operator on H^2 and a simple calculation shows that $C_{\alpha+\beta z}$ commutes with $C_0{}^*$. The operator C_0 on $l_+{}^2$ is called the Cesaro operator [**6**, p. 96]. A theorem of Shields and Wallen [**13**] then implies that there is a bounded analytic function F on $\{z: |1 - z| < 1\}$ such that $C_{\alpha+\beta z} = F(C_0{}^*)$, $\sigma(C_{\alpha+\beta z}) = \mathrm{Closure}\{F(z): |1 - z| < 1\}$ and $\|C_{\alpha+\beta z}\| = \sup\{|\lambda|: \lambda \in \sigma(C_{\alpha+\beta z})\}$. Since we obviously must have $F(1/n) = \beta^{n-1}$ for $n = 1, 2, \ldots$; $F(z) = \beta^{(1/z)-1}$ is the required function. Hence

$$\begin{aligned} \sigma(C_{\alpha+\beta z}) &= \mathrm{Closure}\{F(z): |1 - z| < 1\} \\ &= \mathrm{Closure}\{\beta^{(1/z)-1}: |1 - z| < 1\} \\ &= \{\lambda: |\lambda| \leqq \beta^{-\frac{1}{2}}\}. \end{aligned}$$

and

$$\|C_{\alpha+\beta z}\| = \sup\{|\lambda|: \lambda \in \sigma(C_{\alpha+\beta z})\} = \beta^{-\frac{1}{2}}.$$

A theorem of Kriete and Trutt [**8**] states that C_0 is a subnormal operator with a cyclic vector and hence every operator commuting with C_0 is subnormal [**14**]. Thus $C_{\alpha+\beta z}$ is cosubnormal. We remark that $C_{\alpha+\beta z}$ is the adjoint of the Euler summability matrix of order $\alpha/(1 - \alpha)$ [**6**, p. 178]. Thus the spectrum of the Euler matrix of order $\alpha/(1 - \alpha)$ on l^2 is $\{z: |z| \leqq (1 - \alpha)^{-\frac{1}{2}}\}$.

We next consider the case when α is not positive. Then $\alpha = |\alpha|e^{i\theta}$ and $|\alpha| + \beta = 1$. However, it is easily checked using the unitary operator $C_{e^{i\theta}z}$ that $C_{\alpha+\beta z}$ is unitarily equivalent to $C_{|\alpha|+\beta z}$. Hence $C_{\alpha+\beta z}$ is again a cosubnormal operator whose spectrum is the closed disk of radius $\beta^{-\frac{1}{2}}$ centered at the origin.

An alternative proof for 3(iv) would be to first try and prove that $||C_{\alpha+\beta z}|| = \beta^{-(\frac{1}{2})}$ and then notice that $(1 - z)^{(1/\lambda)-1}$ is an eigenvector for $C_{\alpha+\beta z}$ corresponding to the eigenvalue $\beta^{(1/\lambda)-1}$ where $|1 - \lambda| < 1$.

Notice that if $\beta \neq 1$ then $\alpha/(1 - \beta)$ is the only fixed point of $\varphi(z) = \alpha + \beta z$. We remark that the real distinction between 3(iv) and 3(i–iii) is that in 3(iv) the fixed point of φ is on the unit circle, while in 3(i–iii) the fixed point of φ is in D.

Theorem 3(iii) can be generalized in the following manner. If $\varphi \in H^\infty$ maps D into D, define $\varphi_n \in H^\infty$ inductively by $\varphi_1(z) = \varphi(z)$, $\varphi_n(z) = \varphi_{n-1}(\varphi(z))$.

PROPOSITION 3. *Suppose that* $\varphi \in H^\infty$ *maps* D *into* D *and that for some integer* n *there is an* r, $0 < r < 1$, *such that* $|\varphi_n(z)| \leqq r < 1$ *for all* $|z| < 1$. *Then* C_φ^n *is compact. Furthermore, if* φ *has a fixed point* z_0 *in* D *and* $\beta = \varphi'(z_0)$, *then* $\sigma(C_\varphi) = \text{Closure}\{1, \beta, \beta^2, \ldots\}$.

Proof. By Theorem 5.2 in [**12**], $C_\varphi^n = C_{\varphi_n}$ is compact. The last statement follows as in Theorem 3(iii).

Remarks. 4. H. Schwartz in [**12**] proves that if $\varphi \in H^\infty$ maps D into D and has a fixed point z_0 in D and if $\varphi'(z_0) \neq 0$ then $\{\varphi'(z_0)^n\}_{n=0}^\infty$ are eigenvalues for C_φ and these are the only eigenvalues. In Theorem 3(iv) and Theorems 5 and 6 in [**10**] the eigenvalues are related to the fixed points of φ on the unit circle. Is there some general connection between fixed points of φ on the unit circle and eigenvalues for C_φ?

5. Using Schur's test [**5**, p. 22] one can show that $||C_{\alpha+\beta z}|| \leqq (1 - |\alpha|)^{-\frac{1}{2}}$. Is this an equality?

6. Theorem 3(iii) yields perhaps the worst possible example of a noncompact operator T whose square is compact, since T and T^2 possess common simple eigenvectors that span H^2.

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POLYNOMIAL RINGS WITH THE OUTER PRODUCT PROPERTY

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Note. We shall assume throughout that all rings R are noetherian. This property is not used in some of the lemmas but it intercedes before the main theorem. We retain this assumption to ease the exposition.

Introduction. In [**3**] Lissner defined a class of rings called *outer product rings*, (OP-rings). These are commutative rings R with identity for which *every* exterior vector $v \in \bigwedge^{n-1}R^n$ is decomposable, i.e., $v = v_1 \wedge \ldots \wedge v_{n-1}$ with $v_i \in R^n$, $i = 1, \ldots, n-1$.

If we look only at those vectors $v \in \bigwedge^{n-1}R^n$ whose co-ordinates with respect to any basis of $\bigwedge^{n-1}R^n$ generate the unit ideal in R and consider those rings R for which all vectors of this type are decomposable, we obtain the class of rings which have been referred to as *Hermite-rings* (H-rings, see also Lissner [**3**]). This class of H-rings evidently contains the class of OP-rings.

PROPOSITION A. *R is an H-ring if and only if for any n elements of R, $a_1, \ldots, a_n$, such that the ideal they generate in R is R, there exists an invertible $n \times n$ matrix with first row $(a_1, \ldots, a_n)$.*

Proof. The reader should refer to [**3**, § 2, Proposition 2.1 and Corollary] for a proof of this statement.

One of the major reasons for considering H-rings is contained in the following proposition.

PROPOSITION B. *The following two statements are equivalent.*

(i) *R is an H-ring.*

(ii) *If P is a finitely generated projective R-module such that $P \oplus R^m \cong R^s$ for two integers m, s, then $P \cong R^{s-m}$.*

Proof. See [**11**, Proposition 12.2, p. 185].

In [**7**] Serre asked if every f.g. projective R-module, $R = k[X_1, \ldots, X_n]$ where k is a field, is necessarily free. Serre showed [**7**, Proposition 10] that if P is a f.g. projective $k[X_1, \ldots, X_n]$-module then $P \oplus R^m \cong R^s$ for some integers m, s, (depending on P). In view of Proposition B above, Serre's question amounts to asking whether $k[X_1, \ldots, X_n]$ is an H-ring.

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In [**10**], Seshadri showed that for $n = 2$, finitely generated projective $k[X_1, \ldots, X_n]$-modules are indeed free, while in [**4**] Lissner showed that $R[X]$, when R is a principal ideal domain (P.I.D.), is an OP-ring, thereby obtaining Seshadri's theorem as a consequence.

Our aim in this paper is to classify all those noetherian rings R such that $R[X]$ is an OP-ring. This is accomplished via the following theorem.

THEOREM. *Let R be a noetherian ring. The following are equivalent:*

(i) *$R[X]$ is an* OP-*ring.*

(ii) *R is a direct sum of rings of global dim $\leqq 1$ and special principal ideal rings.*

In the second section we completely classify finitely generated projective $R[X]$-modules, when $R[X]$ is an OP-ring.

1. The outer product property.

We recall a few lemmas.

LEMMA 1. *Let R be an* OP-*ring and $\phi : R \rightarrow S$ be a surjection. Then S is an* OP-*ring.*

Proof. See [**13**, Proposition 1.3].

LEMMA 2. *Let R be a ring and let S be a multiplicatively closed subset of R ($0 \notin S$, $1 \in S$). If R is an* OP-*ring then so is R_S.*

Proof. See [**5**, Proposition 4.5].

COROLLARY *Let $R[X]$ be an* OP-*ring and $p \subset R$ be a prime ideal. Then $R_p[X]$ is an* OP-*ring.*

Proof. The proof is clear from Lemma 2.

In view of the corollary to Lemma 2 it is appropriate to begin considering OP-rings of the form $R[X]$ in the case that R is local.

The following lemma is probably well-known, but inasmuch as there appears no proof in the literature we shall include a proof here for completeness.

We recall the following definition.

Definition. Let (R, m) be a local ring. The *v-dimension* (*v-dim*) of R is $= \dim_{R/m} m/m^2$.

It is well-known (see, e.g., [**6**, p. 189]) that the v-dimension of R is equal to the minimal number of elements in a generating set for m.

LEMMA 3. *Let (R, m) be a local ring of v-dimension $=s$. Then there exists a maximal ideal in $A = R[X]$, say p, such that A_p has* v-dim $= s + 1$.

Proof. We suppose $m = (u_1, \ldots, u_s)$ and let $p \subset A$, $p = (u_1, \ldots, u_s, X)$. It is clear that p is a maximal ideal of A. Let $\phi : A \rightarrow A_p$ be the canonical map, $\phi(f(x)) = f(x)/1$. Evidently $\phi(p)A_p$ is the maximal ideal of A_p and is generated by $\{u_1/1, u_2/1, \ldots, u_s/1, x/1\}$. If we can show that *this* set of generators is minimal we will know that v-dimension $A_p = s + 1$, since in a local ring all minimal generating sets have the same number of elements.

Claim 1. $x/1 \notin (u_1/1, u_2/1, \ldots, u_s/1)$ in A_p: Suppose

$$\frac{x}{1} = \frac{f_1}{g_1}\frac{u_1}{1} + \ldots + \frac{f_s}{g_s}\frac{u_s}{1}$$

where $f_i, g_i \in R[X]$, and $g_i \notin p$. Then

$$\frac{x}{1} = \frac{f_1 g_2 \ldots g_s u_1 + g_1 f_2 g_3 \ldots g_s u_2 + \ldots + g_1 g_2 \ldots g_{s-1} f_s u_s}{g_1 g_2 \ldots g_s}.$$

From the definition of equality in A_p, there exists $k \in R[X]$, $k \notin p$, such that

$$(*) \quad k g_1 g_2 \ldots g_s x = k f_1 g_2 \ldots g_s u_1 + k g_1 f_2 g_3 \ldots g_s u_2 + \ldots + k g_1 g_2 \ldots g_{s-1} f_s u_s.$$

Since $g_i \notin p$, the constant term of g_i is *not* in $m \subset R$; similarly for k. Thus, on the left side of equation $(*)$, the coefficient of x is (const. term of k) $\cdot \prod_{i=1}^{s}$ (const. term g_i). Since all these factors are in R but not in m, and m is prime, the coefficient of x on the left side of $(*)$ is an element of R not in m. Evidently all the coefficients on the right side of $(*)$ are in m. This contradiction establishes the claim.

Claim 2. $u_1/1 \notin (u_2/1, \ldots, u_s/1, x/1)$: (We shall observe that the argument given here does not depend on u_1, but would work for any u_i.) Suppose $u_1/1 \in (u_2/1, \ldots, u_s/1, x/1)$; then

$$\frac{u_1}{1} = \frac{f_1}{g_1}\cdot\frac{x}{1} + \frac{f_2}{g_2}\cdot\frac{u_2}{1} + \ldots + \frac{f_s}{g_s}\cdot\frac{u_s}{1}, \; f_i, g_i \in R[X] \quad \text{and} \quad g_i \notin p.$$

Thus,

$$\frac{u_1}{1} = \frac{f_1 g_2 \ldots g_s x + g_1 f_2 g_3 \ldots g_s u_2 + \ldots + g_1 g_2 \ldots g_{s-1} f_s u_s}{g_1 g_2 \ldots g_s}.$$

Again by the definition of equality in A_p, there exists $k \in R[X]$, $k \notin p$ such that

$$(\dagger) \quad u_1 k g_1 g_2 \ldots g_s = k f_1 g_2 \ldots g_s x + k g_1 f_2 g_3 \ldots g_s u_2 + \ldots + k g_1 g_2 \ldots g_{s-1} f_s u_s.$$

As before, the constant term of $k g_1 \ldots g_s$ is in R not in m, call it t_0. Now $t_0 u_1 \neq 0$, since u_1 is part of a minimal generating set for m and $t_0 \notin m$.

The constant term on the right side of $(\dagger)$ is $\neq 0$ and $t_0 u_1 =$ (const. term of $k g_1 f_2 g_3 \ldots g_s$) $\cdot u_2 + \ldots +$ (const. term of $k g_1 g_2 \ldots g_{s-1} f_s$) $\cdot u_s$.

This equation contradicts the minimality of the generating set $\{u_1, \ldots, u_s\}$ of m, and establishes Claim 2.

Thus $\phi(p)A_p$ needs $s + 1$ generators and so v-dimension $A_p = s + 1$.

PROPOSITION 1. *If* (R, m) *is a local ring of v-dimension* $\geqq 2$, *then* $R[X]$ *is not an* OP-*ring*.

Proof. If $R[X]$ is an OP-ring, then for any prime ideal $p \subset R[X]$, $R[X]_p$ is an OP-ring [**5**, Proposition 4.5]. By Lemma 3, there exists maximal ideal $p \subset R[X]$ such that v-dimension $R[X]_p = (v\text{-dimension } R) + 1 > 2$. But local OP-rings must have v-dim $\leqq 2$ [**13**].

Thus, $R[X]$ is not an OP-ring.

PROPOSITION 2. *If R is a ring of Krull dimension $\geqq 2$, then $R[X]$ is not an* OP-*ring.*

Proof. If Krull dimension $R \geqq 2$ there exists a maximal ideal m in R such that Krull dimension $R_m \geqq 2$. Since Krull dimension $R_m \leqq v$-dimension R_m, we see by Proposition 1 that $R_m[X]$ is not an OP-ring. Thus, $R[X]$ is not an OP-ring by Lemma 2.

In view of Proposition 2 we need only consider rings R of Krull Dimension $\leqq 1$.

If Krull dimension $R = 0$ then all prime ideals in R are maximal and R is a finite direct sum of *primary* rings, i.e., rings with exactly one prime ideal, (finite because R is noetherian). We recall the following proposition.

PROPOSITION 3. *If $R = R_1 \oplus \ldots \oplus R_s$ then, R is an* OP-*ring* $\Leftrightarrow$ *R_i is an* OP-*ring for each $i = 1, \ldots, s$.*

Proof. See [**5**, Theorem 5.4].

So, if R has Krull dimension $= 0$, $R = R_1 \oplus \ldots \oplus R_s$, where the R_i are all primary rings and $R[X] = R_1[X] \oplus \ldots \oplus R_s[X]$. By Proposition 3, $R[X]$ is an OP-ring if and only if $R_i[X]$ is an OP-ring for each $i = 1, 2, \ldots, s$.

PROPOSITION 4. *Let R be a primary ring with prime ideal p. $R[X]$ is an* OP-*ring* $\Leftrightarrow$ *R is a special principal ideal ring or R is a field.* (Recall that R is a special principal ideal ring if R has a unique prime ideal, $p = (u)$, which is nilpotent.)

Proof. $\Rightarrow$: Since $R[X]$ is an OP-ring so is R, by Lemma 1. A primary ring is a local ring and so by Proposition 1, v-dim $R < 2$. If v-dimension $R = 0$ then R is a field. So we may assume that v-dimension $R = 1$. In that case $p = (u)$ and p is nilpotent, i.e., R is a special principal ideal ring.

$\Leftarrow$: If R is a field then $R[X]$ is a principal ideal domain and so is an OP-ring [**3**, Theorem 2.2]. If R is a special principal ideal ring then R is a complete local ring and hence by a theorem of I. S. Cohen [**2**], R is the homomorphic image of a regular local ring of Krull dimension equal to the v-dimension of R. Let A be such a regular local ring of Krull dimension $= 1$, and $\phi : A \to R$ the surjection provided by the theorem. Now A is a principal ideal domain and so $A[X]$ is an OP-ring [**4**]. We extend ϕ to a homomorphism, which we will also call ϕ, $\phi : A[X] \to R[X]$. This new ϕ is also a surjection. Since $A[X]$ is an OP-ring, so is $R[X]$.

Thus, if R has Krull dimension 0, $R[X]$ is an OP-ring if and only if R is a finite direct sum of fields and principal ideal rings.

It remains only to consider rings R of Krull dimension $= 1$. We may as well assume that the prime spectra of the rings considered are connected, for if not then R is a finite sum of rings of Krull dimension $\leqq 1$, whose prime spectra are connected. We have already considered the case of Krull dimension zero. Since the OP property is invariant under direct sums we are reduced to considering the problem in one summand.

PROPOSITION 5. *Let R be a ring with connected prime spectrum and Krull dimension one.*

$R[X]$ is an OP-*ring* $\Leftrightarrow$ *R has global dimension one.*

Proof. $\Leftarrow$: If R has global dimension one then R is a direct sum of Dedekind domains [**1**, Proposition 4.13]. Since the prime spectrum of R is connected, R is a Dedekind domain. That $R[X]$ is an OP-ring follows from [**12**, Theorem 1.2].

$\Rightarrow$: Let m be any maximal ideal of R. Suppose that m is also a minimal prime ideal of R. Since R is noetherian there are only finitely many minimal prime ideals of R, say $m, p_1, \ldots, p_s$. In Spec(R) we let

$$V(\mathfrak{A}) = \{\text{prime } q \subset R | \mathfrak{A} \subseteq q\}$$

for any ideal $\mathfrak{A}$. The $V(\mathfrak{A})$ are all closed sets in Spec(R) and Spec$(R) = (\{m\}) \cup (\bigcup_{i=1}^{s} V(p_i))$. Clearly $\{m\} \cap V(p_i) = \phi, i = 1, \ldots, s$. Since $\bigcup_i V(p_i)$ is closed in Spec(R) we have that $\{m\}$ is open. Since $\{m\} = V(m)$ we also have that $\{m\}$ is closed. This is a contradiction since Spec(R) is connected. Thus m is not a minimal prime of R and so the Krull dimension of R_m is 1.

Since $R[X]$ is an OP-ring so is $R_m[X]$ by Lemma 2. Hence, by Proposition 1, v-dimension $R_m < 2$. Since in any event Krull dimension $R_m \leqq v$-dimension R_m, we have both dimensions $= 1$. Hence global dimension $R_m = 1$. Since this maximal ideal was arbitrarily chosen and since global dimension $R = \sup$ global dimension R_m, (m varying over the maximal ideals of R) we have global dimension $R = 1$.

Thus, if R has Krull dimension 1 and $R[X]$ is an OP-ring then R is a finite sum of rings of global dimension 1, special PIR's and fields. This, together with our previous results proves:

THEOREM 1. *Let R be a commutative noetherian ring with* 1. *$R[X]$ is an* OP-*ring* $\Leftrightarrow$ *R is a direct sum of rings of global dimension* $\leqq 1$ *and special principal ideal rings.*

The following corollary is an immediate consequence.

COROLLARY 1.1. *If R is a domain the following are equivalent.*

(1) *R is a Dedekind domain* (*possibly a field*).

(2) *$R[X]$ is an* OP-*ring.*

(Note: The referee has brought to my attention a recent article by Kleiner in Mat. Sb. *84*, No. 4 (1971), 526–536, in which he also obtains Corollary 1.1 above. Our Theorem 1 then, generalizes his result.)

2. Projective $R[X]$-modules when $R[X]$ is an OP-ring. In view of Theorem 1 of the previous section if one wishes to consider the structure of finitely generated projective $R[X]$-modules when $R[X]$ is an OP-ring it suffices to consider the structure of finitely generated projective $R[X]$ modules when R is a connected ring of global dimension $\leqq 1$ and when R is a special principal ideal ring.

In view of the theorem of Serre [**8**], we know precisely what the situation is when global dimension of R is $\leqq 1$. If R is a field or a principal ideal domain then all finitely generated projectives are free; while if R is a Dedekind domain, not a field or a principal ideal domain, then all finitely generated projective $R[X]$-modules have the form $(R[X])^s \oplus I$ where I is a projective ideal of $R[X]$.

The only case left to consider is the case when R is a special principal ideal ring.

THEOREM 2. *Let R be a special principal ideal ring. Then finitely generated projective $R[X]$-modules are free.*

Proof. Let $p = (u) \subset R$ be the unique prime ideal of R, and let $p[X] = \{a_0 + a_1x + \ldots + a_nx^n | a_i \in p\}$.

Then p is nilpotent, say $p^m = 0$, so $p[X]^m = 0$ also, and $R[X]$ is therefore trivially complete with respect to the $p[X]$-topology. Proposition 2.29 of [**11**] then applies and

$$K_0(R[X]) \cong K_0\left(\frac{R[X]}{p[X]}\right).$$

Since

$$\frac{R[X]}{p[X]} \cong \frac{R}{p}[X] \quad \text{and} \quad K_0\left(\frac{R}{p}[X]\right) \cong Z,$$

it follows that $K_0(R[X]) \cong Z$, i.e., every finitely generated projective $R[X]$-module has a free complement. Since $R[X]$ is an H-ring we have, by Proposition B, that every finitely generated projective $R[X]$-module is then free.

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THE STRICT TOPOLOGY IN A COMPLETELY REGULAR SETTING: RELATIONS TO TOPOLOGICAL MEASURE THEORY

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1. Introduction. Let X be a locally compact Hausdorff space, and let $C^*(X)$ denote the space of real-valued bounded continuous functions on X. An interesting and important property of the strict topology β on $C^*(X)$ was proved by Buck [**2**]: the dual space of $(C^*(X), \beta)$ has a natural representation as the space of bounded regular Borel measures on X.

Now suppose that X is completely regular (all topological spaces are assumed to be Hausdorff in this paper). Again it seems natural to seek locally convex topologies on the space $C^*(X)$ whose dual spaces are (*via* the integration pairing) significant classes of measures. Motivated by this idea, Sentilles [**24**] has considered locally convex topologies β_0, β, and β_1 on $C^*(X)$ which yield as dual spaces the tight, τ-additive, and σ-additive Baire measures of Varadarajan [**30**]. If X is locally compact, the topologies β_0 and β coincide and are precisely the original strict topology of Buck.

The topology β_0 on $C^*(X)$ has an intuitively appealing description: it is the finest locally convex topology which, when restricted to sets bounded in the supremum norm, coincides with the compact-open topology. However, it is defective with respect to a desirable property of the dual space: a weak*-compact set of tight measures need not be β_0-equicontinuous. This bears on a question posed by Buck in the locally compact setting: when is $(C^*(X), \beta)$ a Mackey space?

A partial answer can be found in the work of LeCam [**15**] and, independently, Conway [**4**]: if X is σ-compact locally compact, then $(C^*(X), \beta)$ is a strong Mackey space (i.e., weak*-compact subsets of the dual space are β-equicontinuous). We note that the results of Conway are actually somewhat stronger than this: "weak*-compact" can be replaced in the previous sentence by "weak*-countably compact", and "σ-compact" can be replaced by "paracompact".

The first modification will not be employed here for the sake of simplicity; the second will not be emphasized because it relies on the special decomposition property of paracompact locally compact spaces [**6**, p. 241].

Using the Conway-LeCam result, Sentilles showed that $(C^*(X), \beta_1)$ is always a strong Mackey space. The difficulty with β_1 is its definition, which is

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phrased in terms of the Stone-Čech compactification βX of a given space X and has not yet been described in terms of convergence of nets or sequences of continuous functions on X.

Spaces X for which β_0, β, and β_1 coincide on $C^*(X)$ thus possess the virtues of both β_0 and β_1 and avoid their disadvantages. We refer to such spaces as *β-simple*, and investigate topological conditions on X which are related to β-simplicity. A space is β-simple if and only if (1) every σ-additive measure is tight and (2) every weak*-compact set of positive tight measures is β_0-equicontinuous (equivalently, a tight set of measures).

Spaces for which (1) holds have been investigated by Knowles [**14**] and Moran [**18**]; the latter refers to such spaces as "strongly measure-compact". To their results we add only the observations that the class of strongly measure-compact spaces is invariant under the formation of closed subspaces and countable intersections of subspaces of a fixed space and contains all the σ-compact spaces.

Spaces satisfying (2) have been called T-spaces [**24**] or Prohorov spaces. Not every space is a T-space, but locally compact spaces and complete metric spaces are known to have this property. Relying on an interesting characterization of weak*-compactness due to Topsøe [**27**], we obtain a number of permanence properties. For example, the class of T-spaces is preserved by taking closed subspaces, countable products, and countable intersections of subspaces of a fixed space. An open subset of a T-space is a T-space; conversely, if X is a union of open T-subspaces, then X is a T-space. The last result holds for a finite union of closed T-subspaces, but may fail for a countable union. A result concerning preservation of the T-space property under continuous maps is given, and a number of relevant counter-examples are recorded.

Next we prove an extension of the Conway-LeCam Theorem for β_0. A space is hemicompact if every compact subset is contained in some member of a fixed countable family of compact subsets; a k-space if the topology is determined by the compact subsets [**6**, p. 248]. Then we have: every hemicompact k-space is a T-space (and is, indeed, β-simple). More generally, if X is a topological sum of hemicompact k-spaces, then $(C^*(X), \beta_0)$ is a strong Mackey space.

Recent results of Preiss [**22**] show that the analogous assertions for σ-compact metric spaces fail; indeed such a space is a T-space if and only if it admits a complete metric. Here it is shown that when X is σ-compact metric, then $(C^*(X), \beta_0)$ is a Mackey (or strong Mackey) space if and only if X admits a certain decomposition into σ-compact locally compact subspaces.

Combining the results on strongly measure-compact spaces and T-spaces, we state a general theorem on preservation of β-simplicity. Also, an example is given of a β-simple space for which $\beta_0 = \beta = \beta_1$ is not complete (or even sequentially complete).

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2. Notation and preliminary results. Basic references for topology and topological vector spaces are, respectively, Dugundji (**6**) and Schaefer [**23**]. We recall a few basic concepts and results from topological measure theory as given by Varadarajan [**30**]. For each completely regular space X, let $M(X)$ be the space of continuous linear functionals on $C^*(X)$ (endowed with the usual sup-norm topology).

Definition 2.1. If $\Phi \in M(X)$, then Φ is

(a) *σ-additive*, if for any sequence (f_n) in $C^*(X)$ which is monotone decreasing and pointwise convergent to $0(f_n \downarrow 0)$, $\Phi(f_n) \to 0$.

(b) *τ-additive*, if for any net (f_α) in $C^*(X)$ which is monotone decreasing ($\alpha \leqq \beta$ implies $f_\beta(x) \leqq f_\alpha(x)$ for all $x \in X$) and pointwise convergent to $0(f_\alpha \downarrow 0)$, $\Phi(f_\alpha) \to 0$.

(c) *tight*, if for any uniformly bounded net (f_α) in $C^*(X)$ which converges to 0 uniformly on compact subsets of X, $\Phi(f_\alpha) \to 0$.

The spaces of σ-additive, τ-additive, and tight functionals are written $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$, respectively; the positive functionals in these classes are $M_\sigma^+(X)$, $M_\tau^+(X)$, and $M_t^+(X)$. We have $M_t(X) \subset M_\tau(X) \subset M_\sigma(X) \subset M(X)$. Conditions for equality of these classes have been investigated by Varadarajan, Kirk [**12**; **13**], Knowles [**14**], Moran [**17**; **18**; **19**], and others; we adopt Moran's terminology.

Definition 2.2. A completely regular space X is *measure-compact* if $M_\sigma(X) = M_\tau(X)$; *strongly measure-compact* if $M_\sigma(X) = M_t(X)$.

Every Lindelöf space is measure-compact [**30**, p. 175].

Now $M_\sigma(X)$, $M_\tau(X)$, and $M_t(X)$ are closed subspaces of the Banach space $M(X)$ (with the natural dual norm). More often we shall be interested in the weak*-topology on $M(X)$ or its subspaces with respect to $C^*(X)$. An assertion such as "A is relatively weak*-compact in $M_t^+(X)$" means "the weak*-closure of A *in* $M_t^+(X)$ is weak*-compact".

A fundamental result of the theory is the natural 1-1 correspondence between functionals in $M(X)$ and signed Baire measures on X [**30**, p. 165]. The family $\mathrm{Ba}(X)$ of Baire sets is here defined to be the least σ-algebra containing all zero-sets (sets of the form $f^{-1}(0)$, where $f \in C^*(X)$). A cozero set is the complement of a zero set. A countably additive positive Baire measure μ is zero-set regular: i.e. for $A \in \mathrm{Ba}(X)$,

$$\mu A = \sup\{\mu Z : Z \text{ a zero-set}, Z \subset A\} = \inf\{\mu U : U \text{ a cozero set}, A \subset U\}.$$

A Baire measure μ may be classified as σ-additive, τ-additive, or tight according as the functional

$$\Phi(f) = \int_X f d\mu \quad (f \in C^*(X))$$

has the properties. We shall not distinguish between the functional and the associated measure. There is one further point to consider. It can be deduced from [**14**, p. 144] and [**30**, p. 180] that every tight Baire measure μ has a unique extension to a compact-regular Borel measure ν (the Borel sets are the least σ-algebra containing the open sets). Then μ and ν integrate continuous functions the same, and since we are only concerned with the duality between $C^*(X)$ and $M(X)$, we shall frequently identify $M_t(X)$ with the space of such Borel measures.

Definition 2.3. A subset A of $M_t(X)$ is *tight* if

(a) A is norm-bounded $(\sup\{|\nu|(X):\nu \in A\} < \infty)$.

(b) For every $\epsilon > 0$, there is a compact subset K of X such that $|\nu|(X\backslash K) < \epsilon$ for all $\nu \in A$.

Here $|\nu|$ is, as usual, the total variation of the Borel measure ν.

If $X \subset Y$, then X is absolutely Borel measurable in Y if, for each positive compact-regular Borel measure ν on Y, there are Borel sets A and B in Y with $A \subset X \subset B$ and $\nu(B\backslash A) = 0$.

PROPOSITION 2.4 [**14**, p. 148]. *$M_\tau(X) = M_t(X)$ if and only if X is absolutely Borel measurable in βX.*

PROPOSITION 2.5 [**30**, p. 224]. *If X is a separable metric space, then $M_\sigma(X) = M_t(X)$ if and only if X is absolutely Borel measurable in its completion.*

Metric spaces such that $M_\tau(X) = M_t(X)$ are called inner regular by Dudley [**5**]. By a complete metric space we mean a completely regular space which admits a compatible complete metric. It is well known that such a space is a G_δ set in its Stone-Čech compactification [**32**, p. 180].

COROLLARY 2.6. *Locally compact spaces and complete metric spaces satisfy $M_\tau(X) = M_t(X)$.*

Next we summarize the basic properties of β_0, β, and β_1.

Definition 2.7. The topology β_0 on $C^*(X)$ is the finest locally convex topology which coincides with the compact-open topology on norm-bounded sets.

This concept was mentioned briefly in the remarkable article of LeCam [**15**, p. 217]; it has subsequently been formulated in various equivalent ways by other authors [**8**; **9**; **11**; **25**; **29**]. A synthesis of these efforts is given by Sentilles [**24**]. One useful consequence: a linear map from $C^*(X)$ to a locally convex space E is β_0-continuous if and only if its restriction to each norm-bounded set is continuous with respect to the compact-open topology. Some additional known results about β_0 are contained in the next theorem.

PROPOSITION 2.8. (a) *The dual space of $(C^*(X), \beta_0)$ is $M_t(X)$. A subset of $M_t(X)$ is β_0-equicontinuous if and only if it is tight. Moreover, β_0 is the topology of uniform convergence on tight subsets of $M_t^+(X)$.*

(b) *If X is locally compact, then β_0 coincides with the usual strict topology.*

We refer the reader to Sentilles' paper for the definition of β and β_1 in the completely regular case, and the following results.

PROPOSITION 2.9. (a) *The dual space of* $(C^*(X), \beta_1)$ (*respectively* $(C^*(X), \beta)$) *is* $M_\sigma(X)$ (*respectively* $M_\tau(X)$).

(b) β_1 (*respectively* β) *is the topology of uniform convergence on weak*-compact subsets of* $M_\sigma^+(X)$ (*respectively* $M_\tau^+(X)$).

(c) $(C^*(X), \beta_1)$ *is a strong Mackey space for any* X.

(d) $\beta = \beta_1$ *if and only if* X *is measure-compact.*

(e) *In general,* $\beta_0 \leqq \beta \leqq \beta_1$.

As pointed out by Sentilles, the topologies β and β_1 are equivalent to the topologies T_τ and T_σ introduced by Fremlin, Garling, and Haydon [**8**]. A subset H of $M_\tau(X)$ is *uniformly* τ*-additive* if, whenever $f_\alpha \downarrow 0$ in $C^*(X)$, then $|\mu|(f_\alpha) \to 0$ uniformly with respect to $\mu \in H$; replacing nets by sequences, we have the notion of *uniform* σ*-additivity*. Then from results in [**24**] we have the characterizations of the β- and β_1-equicontinuous sets as the uniformly τ-additive and uniformly σ-additive subsets of M_τ and M_σ, respectively.

Definition 2.10. A space X is a *T-space* if every weak*-compact subset of $M_t^+(X)$ is tight.

It is well-known that locally compact spaces and complete metric spaces are T-spaces.

PROPOSITION 2.11. *If* X *is strongly measure-compact, then the following are equivalent*:

(a) X *is a* T*-space.*

(b) β_0 *is strong Mackey.*

(c) β_0 *is Mackey.*

Proof. (a) $\Rightarrow$ (b): We have $M_\sigma(X) = M_t(X)$, and from 2.8 and 2.10, β_0 is the topology of uniform convergence on weak*-compact subsets of $M_t^+(X)$. Thus $\beta_0 = \beta_1$ (2.9 (b)); hence β_0 is strong Mackey (2.9 (c)).

(b) $\Rightarrow$ (c): This is obvious.

(c) $\Rightarrow$ (a): β_0 and β_1 have the same dual, β_0 is Mackey, and $\beta_0 \leqq \beta_1$. Thus $\beta_0 = \beta_1$ is the topology of uniform convergence on weak*-compact subsets of $M_\sigma^+(X) = M_t^+(X)$. These sets are consequently β_0-equicontinuous and so are tight (2.8 (a)).

Now we can make our central definition.

Definition 2.12. A space X is β*-simple* if $\beta_0 = \beta = \beta_1$.

Sentilles notes that complete separable metric spaces and σ-compact locally compact spaces have this property.

THEOREM 2.13. *The following conditions on a space* X *are equivalent*:

(a) X *is* β*-simple.*

(b) X *is a strongly measure-compact* T*-space.*

(c) $M_\sigma(X) = M_t(X)$, *and* $(C^*(X), \beta_0)$ *is a Mackey space.*

(d) $M_\sigma(X) = M_t(X)$, *and* $(C^*(X), \beta_0)$ *is a strong Mackey space.*

(e) *Every weak*-compact subset of* $M_\sigma(X)$ *is a tight subset of* $M_t(X)$.

(f) *Every weak*-compact subset of* $M_\sigma^+(X)$ *is a tight subset of* $M_t^+(X)$.

Proof. (a) $\Rightarrow$ (b): $\beta_0 = \beta_1$, so the dual spaces coincide. Moreover β_0 is strong Mackey, so X is a T-space (2.11).

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d): These follow from 2.11.

(d) $\Rightarrow$ (e): See 2.8 (a). (e) $\Rightarrow$ (f): The proof is trivial. (f) $\Rightarrow$ (a): We always have $\beta_0 \leqq \beta_1$; but the hypothesis and 2.9 (b) imply that $\beta_1 \leqq \beta_0$.

In view of this result, it seems desirable to investigate the nature of β-simple spaces by considering the problems of characterizing strongly measure-compact spaces and T-spaces separately. However, it is important to note that the conclusion, "β_0 is Mackey", need not follow from either of these properties alone. The space of ordinals less than the first uncountable ordinal is a T-space (since locally compact), but, as Conway [**4**] has shown, $\beta = \beta_0$ is not Mackey. An example of a countable, strongly measure-compact space such that β_0 is not Mackey will be given later (5.6).

3. Measure-compact and strongly measure-compact spaces.

We begin the section with a result on induced mappings of functions and measures; its measure-theoretic content is well-known (e.g., [**18**, p. 495]). If $T: (C^*(Y), \beta)) \to (C^*(X), \beta)$ is continuous, then T is said to be β-β continuous. The observations concerning β_1-β_1 and β-β continuity are due to Judy McKinney.

THEOREM 3.1. *Let* $\varphi: X \to Y$ *be continuous,* $T: C^*(Y) \to C^*(X)$ *be the induced linear mapping defined by* $T(f) = f \circ \varphi$, *and* T^* *be its algebraic adjoint. Then*

(a) $T^*(M'(X)) \subset M'(Y)$, *where* $M' = M, M_\sigma, M_\tau$, *or* M_t.

(b) T *is* β_1-β_1, β-β, *and* β_0-β_0 *continuous.*

(c) *If* $\mu \in M_\sigma(X)$ *and* A *is a Baire set in* Y, *then* $(T^*\mu)(A) = \mu(\varphi^{-1}(A))$.

Proof. (a): The map T is norm-decreasing, hence T^* must map the norm dual of $C^*(X)$ into the norm dual of $C^*(Y)$; moreover, $T^*: M(X) \to M(Y)$ is weak*-continuous. If $\mu \in M_\sigma(X)$ and (f_n) is a sequence in $C^*(Y)$ which is monotone decreasing and pointwise convergent to 0, then $(T(f_n))$ has the same properties. Hence $(T^*\mu)(f_n) = \mu(T(f_n)) \to 0$, and so $T^*\mu \in M_\sigma(Y)$. The proof that $T^*(M_\tau(X)) \subset M_\tau(Y)$ is similar. If (f_α) is uniformly bounded in $C^*(Y)$ and converges to 0 uniformly on compact sets, then (Tf_α) has the same properties. It follows that $T^*(M_t(X)) \subset M_t(Y)$.

(b): Obviously T is a positive linear map; thus if A is a weak*-compact subset of $M_\sigma^+(X)$, then T^*A is a weak*-compact subset of $M_\sigma^+(Y)$. An application of 2.9(b) and standard arguments reveal that T is β_1-β_1 continuous. The proof of β-β continuity is similar. The argument in (a) shows that T, when restricted to norm-bounded sets, is continuous with respect to the compact-open topology. The remark following 2.7 establishes the β_0-β_0 continuity of T.

(c) The proof is omitted, since it follows from known results on measure-preserving transformations; see for example, p. 163 of Halmos' text on measure theory.

In connection with (c), we remark that the stated result is also valid for compact-regular Borel measures and Borel sets.

Moran [**18**] has shown that a Baire set in a measure-compact (respectively strongly measure-compact) space is measure-compact (respectively strongly measure-compact). Using similar arguments, we establish an analogous result for closed subspaces. Recall that the support of a Baire measure μ on X is $\{x \in X$: if U is a cozero set and $x \in U$, then $\mu(U) > 0\}$. A space X is measure-compact if and only if every non-zero member of $M_\sigma^+(X)$ has non-empty support [**17**, p. 634].

Proposition 3.2. *A closed subspace of a measure-compact* (*strongly measure-compact*) *space is measure-compact* (*strongly measure-compact*).

Proof. Let F be a closed subspace of X, with $\varphi : F \to X$ the natural embedding, and define T and T^* as in 3.1. If $\mu \in M_\sigma^+(F)$, then $T^*\mu \in M_\sigma^+(X)$ and, for any Baire set B in X, $(T^*\mu)(B) = \mu(B \cap F)$.

(a) If X is measure-compact and μ is a measure in $M_\sigma^+(F)$ with empty support, then $T^*\mu \in M_\sigma^+(X)$ has empty support. Indeed if $x \in X\backslash F$, then there is a cozero set V in X with $x \in V \subset X\backslash F$, and $(T^*\mu)(V) = 0$. If $x \in F$, then there is a cozero set U in F with $x \in U$ and $\mu(U) = 0$. Thus there is a cozero set W in X with $x \in W \cap F \subset U$, so that $(T^*\mu)(W) = 0$. Since X is measure-compact, $T^*\mu = 0$, and so $\mu = 0$; thus F is measure-compact.

(b) If X is strongly measure-compact, and μ is a non-zero member of $M_\sigma^+(F)$, then there is a compact subset K of X with $\inf\{(T^*\mu)(U) : U$ a cozero set in $X, K \subset U\} = \delta > 0$ [**18**, p. 499]. Let $K_1 = K \cap F$, and suppose V is a cozero set in F with $K_1 \subset V$. Then the closed set $F\backslash V$ and the compact set K are disjoint, hence there is a cozero set W in X with $K \subset W$ and $W \cap (F\backslash V) = \emptyset$. Then $\mu V \geqq \mu(W \cap F) = (T^*\mu)(W) \geqq \delta$. It now follows from Definition 4.1 and Theorem 4.4 of Moran [**18**] that F is strongly measure-compact.

We need one additional result of Moran [**18**, p. 503] : a countable product of strongly measure-compact spaces is strongly measure-compact.

Proposition 3.3. *The intersection of a countable family of strongly measure-compact subspaces of a fixed space is strongly measure-compact.*

Proof. This can be shown directly, but instead we appeal to a simple observation of Negrepontis [**20**, p. 604] : a topological property which is preserved by closed subspaces and countable products is also preserved by countable intersections.

A proof of this result in the locally compact case has been given by Kirk [**12**, p. 340].

Any σ-compact space X is Lindelöf (so $M_\sigma(X) = M_\tau(X)$) and a Borel set in βX (so $M_\tau(X) = M_t(X)$). Thus we have

Proposition 3.4. *A σ-compact space is strongly measure-compact.*

Example 3.5. An arbitrary intersection of strongly measure-compact subspaces of a fixed space need not even be measure-compact. Moran [**17**, p. 638] has given an example of a realcompact space X which is not measure-compact. Such a space is the intersection of a family of σ-compact (locally compact) subspaces of βX [**10**, 8*B*].

4. *T*-spaces. A useful criterion for weak*-compactness of subsets of $M_t^+(X)$ has been obtained by Topsøe [**27**, p. 203]. A family $\mathscr{G}$ of open subsets of X is said to dominate the family $\mathscr{C}$ of compact subsets (in symbols $\mathscr{G} > \mathscr{C}$) provided that each member of $\mathscr{C}$ is a subset of some member of $\mathscr{G}$. We now restate two of Topsøe's results in a form suitable to our needs.

Proposition 4.1. *A subset H of $M_t^+(X)$ is relatively weak*-compact if and only if*

(a) *H is norm-bounded and*

(b) *whenever $\mathscr{G}$ is a family of open subsets of X such that $\mathscr{G} > \mathscr{C}$, for each $\epsilon > 0$ there is a finite subfamily $\mathscr{G}_0$ of $\mathscr{G}$ such that* $\inf\{\mu(X\backslash G) : G \in \mathscr{G}_0\} < \epsilon$ for all $\mu \in H$.

Again we are interpreting members of $M_t^+(X)$ as Borel measures.

Corollary 4.2. *If H is relatively weak*-compact in $M_t^+(X)$ and $\mathscr{G}$ is an open cover of X such that $G_1, G_2 \in \mathscr{G}$ implies the existence of $G_3 \in \mathscr{G}$ with $G_1 \cup G_2 \subset G_3$, then for each $\epsilon > 0$ there is a member G of $\mathscr{G}$ with $\mu(X\backslash G) < \epsilon$* for all $\mu \in H$.

Now we give the main results of the section.

Proposition 4.3. *Let $\varphi : X \to Y$ be a continuous map such that the inverse image of a compact set is compact. If Y is a T-space, then so is X.*

Proof. Let H be weak*-compact in $M_t^+(X)$. Using the notation and results of 3.1, $T^*(H)$ is weak*-compact in $M_t^+(Y)$. Given $\epsilon > 0$, find a compact subset K of Y such that $(T^*\mu)(Y\backslash K) < \epsilon$ for all $\mu \in H$. Then $\mu(X\backslash\varphi^{-1}(K)) < \epsilon$ for $\mu \in H$, and so H is tight.

Corollary 4.4. *A closed subset of a T-space is a T-space.*

Proof. If X is a closed subset of a T-space Y, then the identity map of X into Y satisfies the conditions of the previous theorem.

If C is a Borel set in X, and μ is a compact-regular Borel measure on X, then μ_C, the restriction of μ to the Borel sets of C, is a compact-regular Borel measure on C.

THEOREM 4.5. *Let C be a closed subset of X, H a relatively weak*-compact subset of $M_t^+(X)$. Then $H_1 = \{\mu_C : \mu \in H\}$ is a relatively weak*-compact subset of $M_t^+(C)$.*

Proof. Let $\mathcal{G}$ be a family of open subsets of C which dominates the family of compact subsets of C. For each $G \in \mathcal{G}$, choose an open set O_G in X such that $O_G \cap C = G$. Now let $G' = O_G \cup (X - C)$; then $\mathcal{G}' = \{G' : G \in \mathcal{G}\}$ is a family of open sets which dominates the compact subsets of X.

Given $\epsilon > 0$, Proposition 4.1 lets us choose $G_1', \ldots, G_n'$ in $\mathcal{G}'$ such that, for each $\mu \in H$, $\inf\{\mu(X - G_i')\} < \epsilon$. Thus for each $\mu_C \in H_1$, $\inf\{\mu_C(C\backslash G_i)\} < \epsilon$. The result follows from 4.1.

COROLLARY 4.6. *A space which is a finite union of closed T-subspaces is a T-space.*

Proof. Let $X = \bigcup_1^n C_i$, with each C_i closed in X and a T-space. If H is weak*-compact in $M_t^+(X)$, then $H_i = \{\mu_{C_i} : \mu \in H\}$ is relatively weak*-compact in $M_t^+(C_i)$. Given $\epsilon > 0$, choose a compact subset K_i of C_i with $\mu_{C_i}(C_i\backslash K_i) = \mu(C_i\backslash K_i) < \epsilon/n$ for each i and μ. Let $K = \bigcup_1^n K_i$; then $\mu(X\backslash K) \leqq \sum_1^n \mu(C_i\backslash K_i) < \epsilon$ for all $\mu \in H$.

THEOREM 4.7. *A space which is covered by a family of open T-subspaces is a T-space.*

Proof. Let $\{O_\lambda\}$ be an open cover of X, and assume that each O_λ is a T-space. Then $\{O_\lambda\}$ has an open refinement $\{U_\beta\}$ such that each $\mathrm{Cl}_X U_\beta$ (the closure of U_β in X) is contained in some O_λ; thus each $\mathrm{Cl}_X U_\beta$ is a T-space, by 4.4. Now let $\{V_\alpha\}$ be the family of all finite unions of sets U_β; $\{V_\alpha\}$ is directed upward by inclusion and covers X.

If H is weak*-compact in $M_t^+(X)$, then given $\epsilon > 0$ there is an index α_0 such that $\mu(X\backslash V_{\alpha_0}) < \epsilon/2$ for all $\mu \in H$, by 4.2. Hence $\mu(X\backslash \mathrm{Cl}_X V_{\alpha_0}) < \epsilon/2$ for $\mu \in H$. Now if $V_{\alpha_0} = \bigcup_1^n U_{\beta_i}$, then $\mathrm{Cl}_X V_{\alpha_0} = \bigcup_1^n \mathrm{Cl}_X U_{\beta_i}$ is a T-space, by 4.6. Let $C = \mathrm{Cl}_X V_{\alpha_0}$. Then $H_1 = \{\mu_C : \mu \in H\}$ is relatively weak*-compact in $M_t^+(C)$, by 4.5, and so there is a compact subset K of C with $\mu(C\backslash K) = \mu_C(C\backslash K) < \epsilon/2$ for $\mu \in H$. Then $\mu(X\backslash K) < \epsilon$ for $\mu \in H$. This completes the proof.

COROLLARY 4.8. *A space which is covered by the interiors of a family of closed T-subspaces is a T-space.*

COROLLARY 4.9. *An open subset of a T-space is a T-space.*

Proof. This is immediate from 4.4 and 4.8.

A space is a local T-space if each point has a neighbourhood (not necessarily open) which is a T-space.

COROLLARY 4.10. *A local T-space is a T-space.*

Proof. Let X be a local T-space, x a point in X, N its T-space neighbourhood. Then there is an open subset U of X with $x \in U \subset N$. Now U is a T-space by 4.9; hence X is a T-space by 4.7.

As a trivial consequence, any locally compact space is a T-space (of course this is easy to show directly).

A space which is a countable union of closed T-subspaces need not be a T-space (Example 4.18).

PROPOSITION 4.11. *A space with a locally finite cover of closed T-subspaces is a T-space.*

Proof. Let (C_α) be such a cover of a space X. If $x \in X$, then there is an open set U and sets $C_{\alpha_1}, \ldots, C_{\alpha_n}$ with $x \in U \subset \bigcup_1^n C_{\alpha_i}$. Then U is a T-space by 4.6 and 4.9. Hence X is a T-space by 4.7.

THEOREM 4.12. *A countable product of T-spaces is a T-space.*

Proof. Let (X_n) be a countable family of T-spaces, and let $X = \Pi X_n$, with projections $\varphi_n : X \to X_n$ and induced maps $T_n^* : M_t(X) \to M_t(X_n)$. Let H be weak*-compact in $M_t^+(X_n)$. Given $\epsilon > 0$, for each n there is a compact subset K_n of X_n with $T^*\nu(X_n \backslash K_n) < \epsilon/2^n$ for all $\nu \in H$. Then if $K = \Pi K_n \subset X$, $\nu(X\backslash K) < \epsilon$ for $\nu \in H$.

THEOREM 4.13. *If (X_n) is a countable family of T-subspaces of a fixed space X, then $\bigcap_1^\infty X_n$ is a T-space.*

Proof. The class of T-spaces is preserved under the formation of closed subspaces and countable products. As in Proposition 3.3, the result is immediate.

COROLLARY 4.14. *A G_δ set in a T-space is a T-space.*

Thus Čech complete spaces (spaces which are G_δ's in their Stone-Čech compactifications) are T-spaces. In particular, we have the well-known result (Prohorov's theorem) that every complete metric space is a T-space. Also any space X such that $\beta X \backslash X$ is countable is a T-space.

It has come to our attention that proofs of 4.4, 4.12, and 4.13 are given in [**11**], while other results in this section are improvements on theorems in that paper.

The easiest way to produce a non-trivial weak*-compact set of measures is to find a weak*-convergent sequence. Thus it is of interest to determine if the ranges of weak*-convergent sequences must be tight.

Definition 4.16. A space X is a *sequential T-space* if the range of every weak*-convergent sequence in $M_t^+(X)$ is tight.

LeCam has shown that every metric space is a sequential T-space [**15**, p. 222]. LeCam's result can be extended to the spaces of countable type introduced by Arhangel'skii [**1**]. A space X is of countable type if there are collections (K_α) of compact subsets and $(G_{n,\alpha})$ of open subsets of X such that: (1) any compact subset of X is contained in some K_α and (2) if U is open and contains K_α, then, for some n, $K_\alpha \subset G_{n,\alpha} \subset U$. The sets $G_{n,\alpha}$ may be assumed to be cozero sets. The proof of the next result is very similar to LeCam's argument (see, for example, the monograph of Topsøe [**28**, p. 45]).

THEOREM 4.17. *Every space of countable type is a sequential T-space.*

It is easily seen that 4.3, 4.4, 4.12, and 4.13 remain valid for sequential T-spaces. Also, since any open subset O of a space X can be written as $O = O_1 \cap X$, where O_1 is open in βX (hence locally compact), it follows from 4.13 that 4.9 and consequently 4.14 hold for sequential T-spaces.

We close this section with a series of examples related to the preceding results.

Example 4.18. A σ-compact space need not be a (sequential) T-space. We refer to Varadarajan [**30**, p. 225], Fernique [**7**, p. 24] and Choquet [**3**, p. 14] for three distinct constructions. The first two are hemicompact, and Varadarajan's space, which we refer to below as V, is countable; none is a k-space. Thus an F_σ in a T-space (βV) need not be a T-space.

Example 4.19. If $\varphi : X \rightarrow Y$ is a continuous bijection and a Baire isomorphism (i.e., images and inverse images of Baire sets are Baire sets), then neither φ nor φ^{-1} preserves the T-space property in general. Indeed if $X = V$ and Y is the one-point compactification of the positive integers, the natural one-to-one correspondence φ which leaves the integers fixed serves as one example. On the other hand, Fernique's example of a non-T-space is l^2 (separable Hilbert space) with the weak topology. If we call this space Y, let X be l^2 with the usual norm, and take φ to be the identity map, then X is a T-space (complete metric) but $\varphi(X)$ is not (that φ is a Baire isomorphism follows easily from the fact that every norm-closed ball in l^2 is a zero-set in the weak topology). We remark that measure-compactness and strong measure-compactness are much better behaved under continuous Baire isomorphisms [**18**, 4.6 and 5.2].

Example 4.20. An uncountable product of T-spaces need not be a T-space; an uncountable intersection of T-subspaces of a fixed space need not be a T-space. Indeed the real line $\mathbf{R}$ is a T-space, and V is Lindelöf, hence realcompact [**10**, p. 115], so that V is homeomorphic to a closed subspace of $\mathbf{R}^{C^*(V)}$ [**10**, p. 160]. Thus $\mathbf{R}^{C^*(V)}$ is not a T-space, by 4.4 (note that card$C^*(V) = c$). On the other hand, any space X is an intersection of locally compact (hence T-) subspaces of βX.

Example 4.21. The two most prominent classes of T-spaces, locally compact spaces, and complete metric spaces are Baire spaces [**6**, p. 249], but in general the notions of T-space and Baire space are unrelated. Indeed V is a Baire space which is not a T-space. On the other hand, a linear space E of countably infinite dimension, endowed with the finest locally convex topology [**23**, p. 56] is not a Baire space (it is a countable union of finite-dimensional subspaces, which are closed and nowhere dense). However, E is linearly homeomorphic to the strong dual of the Fréchet-Montel space $\mathbf{R}^N$; hence, by a result of Fernique [**7**, p. 28], E is a T-space.

Some additional examples are given in the next section.

5. T-spaces and k-spaces: an extension of the Conway-LeCam theorem.

Before stating the main result, we consider some topological preliminaries.

A completely regular space X is a k_R-space if the continuity of a real-valued function is implied by its continuity on compact subsets. A k-space is a k_R-space, but the converse is not true (see [**21**] for additional information). We do have the following; it can be deduced from results in [**31**], but we include a proof.

LEMMA 5.1. *A hemicompact k_R-space is a k-space.*

Proof. Let $X = \bigcup_1^\infty K_n$ where (K_n) is an increasing sequence of compact subsets and every compact set is contained in some K_n. Let $D \subset X$, and suppose $D \cap K_n$ is closed for every n. It suffices to show that if $x_0 \in X \backslash D$, then there is an f in $C^*(X)$ with $f|D = 1$, $f(x_0) = 0$. We may assume that $x_0 \in K_1$. There is a function $f_1 \in C^*(K_1)$, $0 \leqq f_1 \leqq 1$, $f_1|(D \cap K_1) = 1$, $f_1(x_0) = 0$. Define f_2' on $K_1 \cup (D \cap K_2)$ by $f_2'|K_1 \equiv f_1$, $f_2'|(D \cap K_2) \equiv 1$. Then f_2' is continuous on the compact subset $K_1 \cup (D \cap K_2)$ of K_2, hence has an extension $f_2 \in C^*(K_2)$, with $0 \leqq f_2 \leqq 1$, $f_2|(D \cap K_2) = 1$. By an obvious induction we can find for each n, $f_n \in C^*(K_n)$, $f_n|(D \cap K_n) \equiv 1$, $f_n|K_{n-1} = f_{n-1}$. The function f defined by $f|K_n = f_n$ is then continuous (by the k_R-property) and separates x_0 and D.

The next result has been essentially obtained by Fremlin, Garling and Haydon [**8**]; the proof given here makes use of Topsøe's criterion (4.1).

THEOREM 5.2. *If X is a hemicompact k_R-space, then $(C^*(X), \beta_0)$ is a strong Mackey space, and X is β-simple.*

Proof. According to 2.11, 2.13, and 3.4, it suffices to show that X is a T-space. Let $X = \bigcup_1^\infty K_n$, where (K_n) is an increasing fundamental family of compact subsets, and let H be weak*-compact in $M_t^+(X)$. If H is not tight, then there exists $\epsilon > 0$, a sequence (μ_n) in H, and compact sets $D_n \subset X \backslash K_n$ such that $\mu_n(D_n) > \epsilon$ for all n. Let $T_n = \bigcup_{i=n}^\infty D_i$; it follows from 5.1 that T_n is closed. If $U_n = X \backslash T_n$, then $K_n \subset U_n$, and so (U_n) dominates the family of compact subsets of X. Thus there is an integer n_0 such that inf $\{\mu(X \backslash U_i) : 1 \leqq i \leqq n_0\} < \epsilon$ for all $\mu \in H$, using 4.1. Since, for any $n \geqq n_0$, $D_n \subset T_i$ for all $i \leqq n_0$, we have a contradiction.

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In order to extend this result to topological sums of spaces, we need a lemma, involving a technique employed by Taylor [**26**] in his study of C^*-algebras. We wish to thank R. A. Fontenot for calling Taylor's result to our attention.

LEMMA 5.3. *Let* $(X_\alpha)_{\alpha \in A}$ *be a family of completely regular spaces with* $X = \sum\{X_\alpha : \alpha \in A\}$ *their topological sum. If* H *is a weak*-compact subset of* $M_\tau(X)$, *then for any* $\epsilon > 0$ *there is a finite subset* F *of* A *such that* $|\mu|(\sum\{X_\alpha : \alpha \in A \backslash F\}) < \epsilon$ for all $\mu \in H$.

Proof. Let $A_0 = \{\alpha \in A : \text{there exists } \mu \in H \text{ with } |\mu|(X_\alpha) > 0\}$. For each $\alpha \in A_0$, choose $f_\alpha \in C^*(X_\alpha)$, $0 \leqq f_\alpha \leqq 1$, and $\mu_\alpha \in H$ such that

$$\int_{X_\alpha} f_\alpha d\mu_\alpha \neq 0.$$

Interpret A as a topological space with the discrete topology, and define $T : C^*(A) \to C^*(X)$ as follows: if $\gamma = (\gamma_\alpha) \in C^*(A)$, then $T(\gamma)|X_\alpha = \gamma_\alpha f_\alpha$. It is easy to see that $T^*(M_\tau(X)) \subset M_\tau(A) = M_t(A) = l_1(A)$; consequently $T^*(H)$ is weakly compact, and therefore norm-compact, in $l^1(A)$. Thus there is a countable subset B_0 of A such that $|T^*\mu|(\{\alpha\}) = 0$ for all $\alpha \in A \backslash B_0$. Since $A_0 \subset B_0$, A_0 is countable.

It follows from τ-additivity that $|\mu|(\sum\{X_\alpha : \alpha \in A \backslash A_0\}) = 0$ for all $\mu \in H$. Moreover, H is weak*-compact in $M_\sigma(X)$; hence if $(f_n) \downarrow 0$ in $C^*(X)$, then $|\mu|(f_n) \to 0$ uniformly with respect to $\mu \in H$ [**8**; **30**]. If A_0 is enumerated as (α_n) and f_n is the characteristic function of $\sum\{X_{\alpha_i} : i \geqq n\}$, the desired result follows.

THEOREM 5.4. *Let* $(X_\alpha)_{\alpha \in R}$ *be a family of completely regular spaces, and suppose that any of the following conditions holds*:

(1) $(C^*(X_\alpha), \beta_0)$ *is Mackey for all* α;

(2) $(C^*(X_\alpha), \beta_0)$ *is strong Mackey for all* α,

(3) $(C^*(X), \beta)$ *is Mackey for all* α;

(4) $(C^*(X), \beta)$ *is strong Mackey for all* α.

If $X = \sum\{X_\alpha : \alpha \in A\}$, *then* $C^*(X)$ *with the corresponding topology has the same property.*

Proof. If H is weak*-compact (convex and circled) in $M_\tau(X)$ or $M_t(x)$, and for any α, we let $H_\alpha = \{\mu | X_\alpha\} \subset M_\tau(X_\alpha)$ (or $M_t(X_\alpha)$), then H_α is weak*-compact (convex and circled). All four results now follow easily upon applying 5.3 and the characterizations of the β- and β_0-equicontinuous sets as the uniformly τ-additive and tight sets, respectively.

COROLLARY 5.5. *The topological sum* X *of a family* $(X_\alpha)_{\alpha \in A}$ *of hemicompact* k*-spaces is a* T*-space, and* $(C^*(X), \beta_0)$ *is a strong Mackey space. However,* X *is* β*-simple if and only if the index set* A *has cardinal of measure zero.*

Proof. The first assertion is an immediate consequence of 5.2 and 5.4. Now each X_α is σ-compact, hence $M_\tau(X_\alpha) = M_t(X_\alpha)$. If 5.3 is applied to a single

τ-additive measure, it is easy to see that $M_\tau(X) = M_t(X)$. Thus from 2.13, X is β-simple if and only if $M_\sigma(X) = M_\tau(X)$. If A is countable, then X is a hemicompact k-space, and therefore is β-simple. If A is uncountable, then the largest cardinal of a closed discrete subset of X is precisely the cardinal of A (any closed discrete subset of X_α is countable). Since X is paracompact, the second assertion is now a consequence of Katetov's theorem [**30**, p. 177].

Example 5.6. The spaces of Varadarajan and Fernique (4.18) are hemicompact spaces, but not T-spaces, and β_0 is not Mackey.

Example 5.7. Every completely regular space can be embedded as a closed subspace of a pseudocompact k_R-space [**21**, p. 56]. Thus a space of this type need not be a T-space.

It is natural to inquire if the preceding results admit a generalization to σ-compact k-spaces, in particular to σ-compact metric spaces. Some remarkable results of D. Preiss [**22**] show that the answer is "no". Indeed we can use 2.11 and the work of Preiss to state:

THEOREM 5.8. *If X is a σ-compact metric space, then the following conditions on X are equivalent:*
(1) *$(C^*(X), \beta_0)$ is Mackey;*
(2) *$(C^*(X), \beta_0)$ is strong Mackey;*
(3) *X is a T-space;*
(4) *X admits a compatible complete metric;*
(5) *X contains no G_δ subspace homeomorphic to the rationals.*

This result is precise, but perhaps slightly deficient in that none of the equivalent properties may be readily determinable for a given space X. The analysis presented here is an attempt to provide a somewhat more computable criterion.

LEMMA 5.9. *If (Y, d) is a complete σ-compact metric space, then Y contains a dense open σ-compact locally compact subspace.*

Proof. Let $y_0 \in Y$, and let $\epsilon > 0$ be given. Let $N(y_0, \epsilon) = \{y : d(y_0, y) < \epsilon\}$. Then the closure D of $N(y_0, \epsilon)$ is a complete σ-compact metric space, and so by the Baire category theorem there is an open set V in Y such that $V \cap D$ is a non-empty, relatively compact subset of D. There is a point $z_0 \in V \cap N(y_0, \epsilon)$, and z_0 has a compact neighbourhood in Y. Thus $Y_0 = \{y \in Y : y$ has a compact neighbourhood$\}$ is the desired subspace (Y_0 is σ-compact because it is separable metric, hence Lindelöf).

THEOREM 5.10. *Let X be a complete σ-compact metric space. Then X has a decomposition into non-empty pairwise disjoint sets $(X_\alpha)_{\alpha<\alpha_0}$, where α_0 is a countable ordinal, such that*
(a) *each X_α is σ-compact locally compact,*
(b) *for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} X_\alpha$ is open in X.*

Proof. Let $X_0 = \{x \in X : x$ has a compact neighbourhood in $X\}$, and, inductively, let $X_\alpha = \{x \in X \backslash \bigcup_{\gamma<\alpha} X_\gamma : x$ has a compact neighbourhood in $X \backslash \bigcup_{\gamma<\alpha} X_\gamma\}$, for each ordinal $\alpha < \omega_1$ (possibly $X_\alpha = \emptyset$ for large α).

According to 5.9, X_0 is a dense open σ-compact locally compact subspace of X, and if $X_0 = X$ we are done. Otherwise, $X \backslash X_0$ is a complete σ-compact metric space, and X_1 is a dense open σ-compact locally compact subspace of $X \backslash X_0$ such that $X_0 \cup X_1$ is open in X. Continuing this process inductively, we obtain a family $(X_\alpha)_{\alpha<\omega_1}$ satisfying (a) and (b).

We claim that, for some $a_0 < \omega_1$, $\bigcup_{\alpha<\alpha_0} X_\alpha = X$. If not, then $X_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. Let $Y = \bigcup_{\alpha<\omega_1} X_\alpha \subset X$. Then Y is open in X, and each $p \in Y$ has an open neighbourhood whose closure in X is a (σ-compact) subset of Y. Thus since Y is Lindelöf, Y is σ-compact. It follows from (b) that, for some $\alpha_0 < \omega_1$, $Y \subset \bigcup_{\beta<\alpha_0} X_\beta$, contradicting the fact that X_{α_0} is non-empty. This completes the proof.

It can be shown that the decomposition given here is maximal in the following sense: if $(Y_\alpha)_{\alpha<\gamma_0}$ is any other decomposition satisfying (a) and (b), then, for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} Y_\alpha \subset \bigcup_{\alpha<\beta} X_\alpha$.

Now we prove the converse to 5.10.

LEMMA 5.11. *If X is a metric space, B is a closed locally compact subspace, and $X \backslash B$ is a T-space, then X is a T-space.*

Proof. Let us first assume that B is compact. Let d be a metric on X, and define $F_1 = \{x : d(x, B) \geqq 1\}$, $F_n = \{x : 1/n \leqq d(x, B) \leqq 1/n - 1\}$ for $n > 1$.

If H is a weak*-compact subset of $M_t^+(X)$, and if H_n is the subset of $M_t^+(F_n)$ consisting of restrictions of members of H, then H_n is relatively weak*-compact (4.5). Since F_n is closed in the T-space $X \backslash B$, it follows from 4.4 that given $\epsilon > 0$, there is a compact set $K_n \subset F_n$ such that $\mu(F_n \backslash K_n) < \epsilon/2^n$ for all n. Now $K_0 = B \cup \bigcup_1^\infty K_n$ is compact (see, for example [**15**, p. 222]), and $\mu(X \backslash K_0) < \epsilon$ for all $\mu \in H$.

Taking up the general case in which B is closed and locally compact, fix $x_0 \in B$, and let V be an open subset of B such that $x_0 \in V$ and $\bar{V}$ is compact. Then $(X \backslash B) \cup \bar{V}$ is a T-space, and so is its open subspace $(X \backslash B) \cup V$, by 4.9. Now $(X \backslash B) \cup V$ is open in X, and it follows from 4.7 that X is a T-space.

THEOREM 5.12. *Let X be a metric space, and suppose X is a disjoint union of non-empty subspaces $(X_\alpha)_{\alpha<\alpha_0}$, where α_0 is an ordinal and*

(a) *each X_α is locally compact,*

(b) *for any $\beta \leqq \alpha_0$, $\bigcup_{\alpha<\beta} X_\alpha$ is open in X.*

Then X is a T-space.

Proof. Note that X need not be σ-compact, and α_0 need not be a countable ordinal. We use transfinite induction on the proposition $P(\gamma)$: every metric space Y with a decomposition $(Y_\alpha)_{\alpha<\gamma}$ into non-empty disjoint subsets satisfying (a) and (b) is a T-space.

Now $P(1)$ is clear, since Y is then locally compact. Suppose $P(\gamma)$ holds for all $\gamma < \alpha_0$, and let $X = \bigcup_{\alpha<\alpha_0} X_\alpha$ satisfy (a) and (b).

Case 1. If α_0 is a limit ordinal, then for each $\gamma < \alpha_0$, $(X_\alpha)_{\alpha<\gamma}$ is a decomposition of $W_\gamma = \bigcup_{\alpha<\gamma} X_\alpha$ which satisfies (a) and (b). Thus W_γ is a T-subspace of X. Moreover, each W_γ is open in X, and so $X = \bigcup_{\gamma<\alpha} W_\gamma$ is a T-space, by 4.7.

Case 2. If $\alpha_0 = \alpha_1 + 1$, then, since $(X_\alpha)_{\alpha<\alpha_1}$ is a decomposition of $W = \bigcup_{\alpha<\alpha_1} X_\alpha$ satisfying (a) and (b), W is a T-space. It now follows from 5.11 that X is a T-space.

COROLLARY 5.13. *If X is a σ-compact metric space, then $(C^*(X), \beta_0)$ is Mackey (or strong Mackey) if and only if X is the union of a pairwise disjoint family $(X_\alpha)_{\alpha<\alpha_0}$ of non-empty σ-compact locally compact subspaces, indexed by some countable ordinal α_0, such that $\bigcup_{\alpha<\beta} X_\alpha$ is open for all $\beta \leqq \alpha_0$.*

Any σ-compact metric space X may be subjected to the canonical decomposition of 5.10. If, at any stage, X_α fails to be dense in

$$X \backslash \bigcup_{\beta<\alpha} X_\beta,$$

then $(C^*(X), \beta_0)$ is not Mackey. In any case, a decision will be reached after countably many iterations.

According to 5.2 and 5.8, any hemicompact metric space admits a complete metric. There is a simpler way to see this, however. It is not difficult to show that a hemicompact metric space is locally compact, hence an open set in its Stone-Čech compactification, and therefore is completely metrizable [**32**, p. 180].

Finally let us note that for any $\alpha < \omega_1$, there is a countable, complete metric space X of rank α: i.e., such that $X_\beta \neq \emptyset$ for $\beta < \alpha$ (notation as in 5.10), while $X_\alpha = \emptyset$. Indeed, this is trivial for $\alpha = 1$. Suppose that, for each $\beta < \alpha$, there is a countable complete metric space $X^{(\beta)}$ of rank β. If α is a limit ordinal, with $\alpha_1 < \alpha_2 < \ldots \to \alpha$, let

$$X^{(\alpha)} = \sum_{n=1}^{\infty} X^{(\alpha_n)}.$$

If $\alpha = \alpha' + 1$, let Y be a countable topological sum of copies of $X^{(\alpha')}$, and let $X^{(\alpha)}$ be a countable topological sum of copies (Y_i) of Y, together with an exceptional point p, having a base of neighbourhoods consisting of sets $\{p\} \cup (\sum_{i=n}^{\infty} Y_i)$. It is not difficult to show that $X^{(\alpha)}$ has the desired properties.

6. Some remarks on the strict topology. Combining 2.13, 3.2, 3.3, 3.4, 4.4, 4.12, 4.13, and 5.2 we have

THEOREM 6.1. *The class of β-simple spaces contains the hemi-compact k-spaces and is preserved by taking closed subspaces, countable products, and countable intersections of subspaces of a fixed space.*

A complete separable metric space (Polish space) is a G_δ set in a compact metric space [**32**, 24.13 and 24B]; but an open set in a compact metric space is a hemicompact k-space.

COROLLARY 6.2. *A Polish space is β-simple.*

If X is locally compact, Buck [**2**] showed that $(C^*(X), \beta_0)$ is a complete locally convex space. Some results on completeness in the completely regular case were given by Sentilles [**24**]. In particular, β_0 is complete if and only if X is a k_R-space.

Example 6.3. A β-simple space such that $\beta_0 = \beta = \beta_1$ is not complete or even sequentially complete: This construction is based on an argument due to Michael [**16**, p. 282]. Let X be the irrationals, and let Y be the space obtained by taking a countable number of copies of the one-point compactification of the positive integers and identifying the points at infinity. Then X is a Polish space and Y is a hemicompact k-space, so $X \times Y$ is β-simple by 6.1 and 6.2. Y may be visualized as a countably infinite collection of "spokes" emerging from a central point p; denote the mth point on the nth spoke by $y(m, n)$, and the union of the first n spokes (including p) by F_n. Let (d_n) be a sequence of rationals increasing monotonically to $\sqrt{2}$, and for each n let $(c_{k,n})$ be a sequence of rationals which satisfies $d_n < c_{k,n} < \sqrt{2}$ for all k and decreases monotonically to d_n.

Let $A = \{(x, y(m, n)) : x$ is irrational, and $d_n < x < c_{m,n}\}$ and let $A_n = A \cap (X \times F_n)$. Since all d_n and $c_{k,n}$ are rational, it is easy to see that each A_n is open and closed in $X \times Y$; hence the characteristic function ψ_n of A_n is continuous. But $(\sqrt{2}, p)$ is a limit point of A, hence the characteristic function ψ_0 of A is not continuous. Now if K is a compact subset of $X \times Y$, then its projection on Y is compact, hence contained in some F_n. It follows that the sequence (ψ_n) is β_0-Cauchy in $C^*(X)$ and converges pointwise to the discontinuous function ψ_0. Thus $(C^*(X), \beta_0)$ is not sequentially complete.

7. An open question.

We do not know any example of a T-space for which $M_t \neq M_\tau$. In this regard it would be of particular interest to determine whether or not the real line with the right half-open interval topology is a T-space.

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RIGHT-ORDERED GROUPS

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1. Introduction. A group G is right-ordered if it can be totally ordered so that for any a, b, c in G, $a < b$ implies that $ac < bc$. Right-ordered groups, considered as order preserving automorphisms of ordered sets, were studied by Cohn in [**4**]; but the first systematic study of the structure of these groups was made by Conrad in [**5**] where he gave several natural characterizations of right-ordered groups. We mention here that the class of right-ordered groups is precisely the subgroup closure of the class of lattice ordered groups (see [**6**], [**7**], [**9**] or [**10**]).

Conrad was particularly successful in the study of the structure of groups G which can be right-ordered in such a way that

(*) for each pair of positive elements a, b in G there exists a positive integer n such that $a^n b > a$.

The class of such groups coincides with the class of groups having a normal system with torsion-free abelian factors. If G is a finitely generated group in this class then G/G' is necessarily infinite. We still do not know whether every finitely generated right-ordered group G has the property that G/G' is infinite. We can only prove the following result.

THEOREM 1. *A finitely generated group $G \neq \{e\}$ can not be right-ordered if G/G' is finite and G has a nilpotent subgroup of finite index in G.*

The above question is significant because it is related to the problem of deciding whether the integral group-ring of a torsion-free group can have zero divisors. For any class $\mathfrak{X}$ of groups, define the class of *locally $\mathfrak{X}$-indicable* groups to consist of those groups G in which every finitely generated non-trivial subgroup has a non-trivial homomorphic image in $\mathfrak{X}$. The terminology is derived essentially from that of Higman [**8**] who proved that the integral group-ring of a locally $\mathfrak{Z}$-indicable group has no zero divisors. (Here $\mathfrak{Z}$ denotes the class of infinite cyclic groups.) It was shown in [**12**] that the integral group-ring of a right-ordered group has no zero divisors. Burns and Hale [**3**, Theorem 2] observed that a locally RO-indicable group is an RO-group, where RO denotes the class of right-ordered groups. In particular locally $\mathfrak{Z}$-indicable groups are RO-groups. We know of the existence of finitely generated torsion-free groups G with G' nilpotent and G/G' finite (see, for example, [**11**, p. 250] or [**2**]). Thus Theorem 1 effectively tells us that a different

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approach is needed to determine whether the integral group-rings of such groups have zero divisors.

Let $\overline{\text{RO}}$ denote the class of those RO-groups in which every right order has the property (*).

THEOREM 2. *Every torsion-free locally nilpotent group is an* $\overline{\text{RO}}$*-group.*

THEOREM 3. *A finitely generated group* $G \neq \{e\}$ *can not be right-ordered if* G/G' *is finite and* G *has a subgroup* K *of finite index in* G *with* $K \in \overline{\text{RO}}$.

It is easy to see that Theorem 1 is a consequence of Theorems 2 and 3. In § 4 we produce an example of a metabelian RO-group that is not an $\overline{\text{RO}}$-group. We do not know if every polycyclic RO-group is an $\overline{\text{RO}}$-group.

We now turn our attention to the following question raised by Ault in [**1**]. Can every partial right-order be extended to a full right-order in a torsion-free nilpotent group? Ault proved the result in the special case when the group is nilpotent of class two. A sub-semigroup P of a group G defines a partial right-order on G if $P \cap P^{-1} = \phi$. The partial right-order is obtained by putting $x < y$ if and only if $yx^{-1} \in P$. A sub-semigroup Q is called an extension of the partial right-order P if $Q \supseteq P$ and $Q \cap Q^{-1} = \phi$, and Q is a (full) right-order if in addition $Q \cup Q^{-1} \cup \{e\} = G$.

THEOREM 4. *Every partial right-order can be extended to a right-order in a torsion-free nilpotent group.*

In § 4 we give an example of a metabelian RO-group in which not every partial right-order can be extended to a right-order.

2. Proofs of Theorems 2 and 3.

LEMMA 2.1 (B. H. Neuman). *Let* G *be a locally nilpotent group,* X *a subset of* G *and* S *the semigroup generated by* X. *Then given* u, v *in* S, *there exists* z, t *in* S *such that* $zu = tv$.

Proof. We use induction on the nilpotency class of $L = \text{group}\langle u, v\rangle$. If L is abelian then take $z = v$ and $t = u$. If L is nilpotent of class $r > 1$, then $M = \text{Group}\langle uv, vu\rangle$ is nilpotent of class $r - 1$ since $vu = uv[v, u]$. Thus there exists a, b in Semigroup$\langle uv, vu\rangle$ satisfying $avu = buv$. Take $z = av$ and $t = bu$.

Let G be a torsion-free locally nilpotent group, P the positive cone of a given right-order on G and a, b in P. Suppose, if possible, that $a^n b < a$ for all integers $n > 0$. Let $S = \text{Semigroup}\langle ab, a\rangle$. By Lemma 2.1, there exists z, t in S such that $zab = ta$. Since a, b are both in P, $S \subseteq P$ so that $t > e$ and $ta > a$. Now $zab = a^{\alpha_1}ba^{\alpha_2}b \ldots a^{\alpha_r}b$ with $\alpha_i > 0$, $i = 1, \ldots, r$. Since $a^n b < a$ for all $n > 0$, $a^{\alpha_1}ba^{\alpha_2}b \ldots a^{\alpha_r}b < a^{\alpha_2+1}b \ldots a^{\alpha_r}b < \ldots < a^{\alpha_r+1}b < a$, a contradiction. This completes the proof of Theorem 2.

We prove Theorem 3 by way of contradiction. Let $<$ be a right-order on G. By hypothesis G is finitely generated torsion-free, G/G' is finite and G has a normal subgroup K of finite index in G with $K \in \overline{\mathrm{RO}}$. Since K is a finitely generated $\overline{\mathrm{RO}}$-group, K/K' contains elements of infinite order. Let I be the isolator of K' in K. Then K/I is torsion-free abelian. Choose coset representatives $e = x_1 < x_2 < \ldots < x_n$ of K in G and take the transfer map $\tau : G \to K/I$ given by

$$\tau(g) = \left(\prod_{i=1}^{n} x_i g(\overline{x_i g})^{-1}\right) I,$$

where $\overline{x_i g}$ is the coset representative of $x_i g$. τ is the trivial map since K/I is torsion-free abelian and G/G' is finite. Observe that if $e < x < y$ then $e < yx^{-1}$. For any $y \geqq x_n$, $y^n \in K$ and $y^n > x_n$. Thus if $g > x_n$ and $g \in K$, then $g^{x_2} \ldots g^{x_n} g > g$ and $g^{x_2} \ldots g^{x_n} g \in I$. Since K is a finitely generated $\overline{\mathrm{RO}}$-group, there exists $g > x_n$ in K such that the convex subgroup generated by g is K (see § 4 of [**5**]). Let C be the union of convex subgroups of K not containing g. Then K/C is isomorphic to a subgroup of the additive group of reals. Hence $I \leqq C$. Also $g > x$ for all $x \in C$. This gives the required contradiction.

3. Proof of Theorem 4. Let G be a torsion-free nilpotent group and let P be the positive cone of a partial right-order $<$ on G. Assume, by way of contradiction, that P is maximal but not a full right-order. Then for some $x \neq e$, $x \notin P \cup P^{-1}$. Since P is maximal we conclude that

$$e \in \mathrm{Semigroup}\langle P, x\rangle \cap \mathrm{Semigroup}\langle P, x^{-1}\rangle.$$

More specifically,

$$x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m = e = x^{-\beta_1} q_1 \ldots x^{-\beta_n} q_n \tag{1}$$

where $p_1, \ldots, p_m, q_1, \ldots, q_n$ lie in P and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ are all positive integers.

If $x \in Z(G)$, the centre of G, then (1) reduces to $x^{\alpha} p = e = x^{-\beta} q$ for some $\alpha, \beta > 0$ and p, q in P. This is not possible since it implies $x^{\alpha\beta} = e$ and $\alpha\beta \neq 0$. Thus $Z(G) \leqq P \cup P^{-1} \cup \{e\}$. Assume that $Z_i(G) \nleqq P \cup P^{-1} \cup \{e\}$ but $z_{i-1}(G) \leqq P \cup P^{-1} \cup \{e\}$ for some integer $i > 1$. ($Z_j(G)$ denotes the jth term of the upper central series of G.) Thus (1) holds for some $x \in Z_i(G) \backslash Z_{i-1}(G)$. We now investigate the consequence of the left-hand side of equation (1).

Let W be the set of all words $x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m$ that are equal to e with α_i's positive, $m \geqq 1$ and p_i's in P. Define a function μ from P to the set of non-negative integers by the rule

$$\mu(p) = \begin{cases} 0 \text{ if } [p, x] = e \\ j \text{ if } [p, x] \in Z_j(G) \backslash Z_{j-1}(G), j = 1, 2, \ldots . \end{cases}$$

For any $w = x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m$ in W, let $|w| = \max\{\mu(p_i) : i = 1, 2, \ldots, m\}$. Note that $|w| = 0$ implies that $[x, p_i] = e$ for $i = 1, \ldots, m$ so that $x^{\alpha_1} p_1 \ldots x^{\alpha_m} p_m = x^{\alpha} p$ with $\alpha = \sum_{i=1}^{n} \alpha_i > 0$ and $p = p_1 \ldots p_m \in P$. Hence $x^{-\alpha} \in P$. We will show that there exists w in W with $|w| = 0$. Suppose, if possible, that $|w| > 0$ for every $w \in W$. Let W_1 be the subset of W consisting of those words w with $|w|$ minimal. We call p_i a ***dominant component*** of $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ if $\mu(p_i) = |w|$. Since $w = e$, there are at least two dominant components in w. Let W_2 be the set of those words in W_1 with the least number of dominant components. Let W_3 be the set of those words $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_2 with $\mu(p_1) = |w|$. Let $j > 1$ be the smallest integer such that for all $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_3, $\mu(p_i) < \mu(p_1)$ for all i satisfying $1 < i < j$, but $\mu(p_j) = \mu(p_1)$ for some w in W_3. Let W_4 be the set of those $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_3 with $\mu(p_j) = \mu(p_1)$. Finally, let W_5 be the set of those $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_4 with m minimal. Of course we are assuming, by way of contradiction, that $m > 1$.

Pick any $w = x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m$ in W_5. Since $x^{\alpha_j}p_j = p_j(x[x, p_j])^{\alpha_j}$, we must have $[x, p_j] < e$, for otherwise

$$x^{\alpha_1}p_1 \ldots x^{\alpha_m}p_m = x^{\alpha_1}p_1 \ldots x^{\alpha_{j-1}}p_{j-1}p_j(x[x, p_j])^{\alpha_j}x^{\alpha_{j+1}} \ldots p_m \equiv w' \in W$$

and $|w| = |w'| = \mu(p_1) \geqq \mu(p'_{j-1})$ where $p'_{j-1} = p_{j-1}p_j$. If $j = 2$ or $\mu(p_1) > \mu(p'_{j-1})$ then we contradict the choice of W_2 and if $j \neq 2$ and $\mu(p_1) = \mu(p'_{j-1})$ then we contradict the choice of W_4. Thus $[p_j, x] > e$ and

$$w_1 = x^{\alpha_1}p_1 \ldots x^{\alpha_{j-1}}p_{j-1}x^{\alpha_j+1}p_{j1}x^{\alpha_{j+1}-1} \ldots p_m \in W_5,$$

where $p_{j1} = p_j[p_j, x] \in P$ and $\mu(p_{j1}) = \mu(p_j)$. Repeated application of the above argument yield w_i, $i = 1, \ldots, \alpha_{j+1}$ where

$$w_i = x^{\alpha_1}p_1 \ldots p_{j-1}x^{\alpha_j+i}p_{ji}x^{\alpha_{j+1}-i} \ldots p_m,$$

$p_{ji} = p_{ji-1}[p_{ji-1}, x] \in P$ and $\mu(p_1) = \mu(p_{ji})$. Now $w_i \in W_5$ for $i < \alpha_{j+1}$, but

$$w_{\alpha_{j+1}} = x^{\alpha_1}p_1 \ldots p_{j-1}x^{\alpha_j+\alpha_{j+1}}p'x^{\alpha_{j+2}} \ldots p_m$$

where $p' = p_{j\alpha_{j+1}}p_{j+1} \in P$ and $\mu(p') \leqq \mu(p_1)$. Now $\mu(p') < \mu(p_1)$ contradicts the choice of W_2 and $\mu(p') = \mu(p_1)$ contradicts the choice of W_5. Note that should $j = m$, we take x^{α_1} for $x^{\alpha_{j+1}}$ and in this case we have

$$w_{\alpha_1} = x^{\alpha_m+\alpha_1}(p_{m\alpha_1}p_1)x^{\alpha_2}p_2 \ldots x^{\alpha_{m-1}}p_{m-1}$$

which contradicts the choice of W_2.

We have thus established that $x^{\alpha}p = e$ for some integer $\alpha > 0$ and some $p \in P$. Similarly we obtain $x^{-\beta}q = e$ for some $\beta > 0$ and some $q \in P$ as a consequence of the right-hand side of (1). These two equations imply that $x^{\alpha\beta} = e$ with $\alpha\beta \neq 0$, a contradiction.

4. Examples.

We give two examples to show the limitations of Theorems 2 and 4.

Let G be the group generated by two permutations, α and τ, of the real line given by:

$$x\alpha = x + 1$$
$$x\tau = x/2.$$

Thus $x(\tau\alpha\tau^{-1}) = x/2(\alpha\tau^{-1}) = ((x+2)/2)\tau^{-1} = x\alpha^2$. In fact it is easy to verify that G is isomorphic to Group$\langle\alpha, \tau; \tau\alpha\tau^{-1} = \alpha^2\rangle$ which is metabelian. G is a subgroup of the group of order-preserving permutations of the real line in the sense that $x < y$ implies $x\theta < y\theta$ for all $\theta \in G$. Thus G is an RO-group and we can order it in the fashion described by Conrad in [**4**], by well-ordering the set $\mathbf{R}$ of real numbers in any appropriate way and then, for any $\theta \in G$, look at the first $r \in \mathbf{R}$ in the well-ordering for which $r\theta \neq r$. Put $\theta > e$ if $r\theta > r$ and $\theta < e$ otherwise. In particular, by well-ordering $\mathbf{R}$ so that 0 is the first element and -1 is the second, we make $\alpha > e$, $\tau > e$ and $\alpha\tau > e$. But $(\alpha\tau)^n\tau(\alpha\tau)^{-1} < e$ for all $n > 0$ since 0 is mapped to $(2^{n-1}/2^n) - 1$ under $(\alpha\tau)^n\alpha^{-1}$. Thus the right-order on G described above does not satisfy the property (*). This example is basically similar to Example III [**5**] of Conrad, except that Conrad's example is more complicated and not metabelian.

Our second example is $G =$ Group$\langle a, b; a^{-1}ba = b^{-1}\rangle$. It is a metacyclic RO-group. $P =$ Semigroup$\langle a^2, b, ba^{-2}\rangle$ defines a partial right-order on G. This is easily verified since $[a^2, b] = e$. Under any extension of P to a full right-order on G we must have $a > e$ since $a^2 \in P$. Also $ba^{-2} \in P$ implies $ba^{-1} > a > e$ and hence $aba^{-1} = b^{-1} > e$, a contradiction.

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SUPER-REFLEXIVE BANACH SPACES

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Introduction. A super-reflexive Banach space is defined to be a Banach space B which has the property that no non-reflexive Banach space is finitely representable in B. Super-reflexivity is invariant under isomorphisms; a Banach space B is super-reflexive if and only if B^* is super-reflexive. This concept has many equivalent formulations, some of which have been studied previously. For example, two necessary and sufficient conditions for super-reflexivity are: (i) There exist positive numbers $\delta < \frac{1}{3}$, A, and r such that $1 < r < \infty$ and $A[\sum |a_i|^r]^{1/r} \leqq ||\sum a_i e_i||$ for every normalized basic sequence $\{e_i\}$ with $\operatorname{char}\{e_i\} \geqq \delta$ and all numbers $\{a_i\}$; (ii) There exist positive numbers $\delta < \frac{1}{2}$, B, and s such that $1 < s < \infty$ and $||\sum a_i e_i|| \leqq B[\sum |a_i|^r]^{1/r}$ for every normalized basic sequence $\{e_i\}$ with $\operatorname{char}\{e_i\} \geqq \delta$ and all numbers $\{a_i\}$.

Definition 1. A normed linear space X being *finitely representable* in a normed linear space Y means that, for each finite-dimensional subspace X_n of X and each number $\lambda > 1$, there is an isomorphism T_n of X_n into Y for which

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \quad \text{if} \quad x \in X_n.$$

Definition 2. A normed linear space X being *crudely finitely representable* in a normed linear space Y means that there is a number $\lambda > 1$ such that, for each finite-dimensional subspace X_n of X, there is an isomorphism T_n of X_n into Y for which

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \quad \text{if} \quad x \in X_n.$$

Definition 3. A *super-reflexive* Banach space is a Banach space B which has the property that no non-reflexive Banach space is finitely representable in B.

It follows directly from known facts that a Banach space is super-reflexive if it is isomorphic to a Banach space that is uniformly non-square [**3**, Lemma C]. Clearly, all super-reflexive spaces are reflexive. The next theorem will enable us to prove easily that super-reflexivity is isomorphically invariant.

THEOREM 1. *A Banach space B is super-reflexive if and only if no non-reflexive Banach space is crudely finitely representable in B.*

Proof. Clearly, a Banach space B is super-reflexive if no non-reflexive Banach space is crudely finitely representable in B. We must show that if a non-reflexive space X is crudely finitely representable in B, then there is a

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non-reflexive space Y that is finitely representable in B. Since X is non-reflexive, there is an $\epsilon > 0$ and a sequence $\{x_n\}$ in the unit ball of X such that

$$\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_k\}, \operatorname{conv}\{x_{k+1}, \ldots\}) > \epsilon$$

for every $k \geqq 1$ [**2**, Theorem 7, p. 114]. Let $\lambda > 1$ be a number such that, for each n, there is an isomorphism T_n of $\operatorname{lin}\{x_1, \ldots, x_n\}$ into B with

$$\lambda^{-1}||x|| \leqq ||T_n(x)|| \leqq \lambda||x|| \text{ if } x \in \operatorname{lin}\{x_1, \ldots, x_n\}.$$

Let $y_i^n = \lambda^{-1}T_n(x_i)$ for $i \leqq n$. Then $||y_i^n|| \leqq 1$ and, if $1 \leqq k < n$,

$$\operatorname{dist}(\operatorname{conv}\{y_1^n, \ldots, y_k^n\}, \operatorname{conv}\{y^n_{k+1}, \ldots, y_n^n\})$$
$$\geqq \lambda^{-2}\operatorname{dist}(\operatorname{conv}\{x_1, \ldots, x_k\}, \operatorname{conv}\{x_{k+1}, \ldots, x_n\}),$$

so that

$$\operatorname{dist}(\operatorname{conv}\{y_1^n, \ldots, y_k^n\}, \operatorname{conv}\{y^n_{k+1}, \ldots, y_n^n\}) \geqq \lambda^{-2}\epsilon.$$

Now the procedure used in the proof of Lemma B in [**3**] gives a space Y that is finitely representable in B and is non-reflexive by virtue of having a sequence $\{\eta_n\}$ for which $||\eta_n|| \leqq 1$ and, for every $k \geqq 1$,

$$\operatorname{dist}(\operatorname{conv}\{\eta_1, \ldots, \eta_k\}, \operatorname{conv}\{\eta_{k+1}, \ldots\}) \geqq \lambda^{-2}\epsilon.$$

THEOREM 2. *Super-reflexivity is invariant under isomorphisms. A Banach space B is super-reflexive if and only if B^* is super-reflexive.*

Proof. It follows from Theorem 1 that super-reflexivity is invariant under isomorphisms. Now suppose that X is non-reflexive and finitely representable in B. Since X^* is non-reflexive, there is an $\epsilon > 0$ and a sequence of linear functionals $\{f_n\}$ in the unit ball of X^* for which

$$\operatorname{dist}(\operatorname{conv}\{f_1, \ldots, f_k\}, \operatorname{conv}\{f_{k+1}, \ldots\}) > \epsilon \quad \text{if} \quad k \geqq 1.$$

For a positive integer n and a finite-dimensional subspace X_p of X, let T map X_p into B as described in Definition 1. Define ϕ_k^n for $k \leqq n$ by letting $\phi_k^n[T(x)] = f_k(x)$ if $x \in X_p$, and then extending ϕ_k to all of B. If X_p is chosen suitably and λ is close enough to 1, then $||\phi_k|| < 2$ and

$$\operatorname{dist}(\operatorname{conv}\{\phi_1^n, \ldots, \phi_k^n\}, \operatorname{conv}\{\phi^n_{k+1}, \ldots, \phi_n^n\}) > \epsilon \tag{1}$$

if $1 \leqq k < n$. Again, the procedure of [**3**, Lemma B] gives a space Y that is finitely representable in B^* and is non-reflexive by virtue of containing a bounded sequence $\{\eta_n\}$ for which

$$\operatorname{dist}(\operatorname{conv}\{\eta_1, \ldots, \eta_k\}, \operatorname{conv}\{\eta_{k+1}, \ldots\}) \geqq \epsilon \quad \text{if} \quad k \geqq 1.$$

Conversely, suppose Y is non-reflexive and finitely representable in B^*. As in the proof of Theorem 1, it then follows that there is an $\epsilon > 0$ such that, for every positive integer n, there is a subset $\{\phi_1^n, \ldots, \phi_n^n\}$ of the unit ball of B^* for which (1) is satisfied. The procedure of [**3**, Lemma B] then gives a space X that is finitely representable in B and is non-reflexive by virtue of there being a bounded sequence of linear functionals $\{f_n\}$ in X^* for which

$$\operatorname{dist}(\operatorname{conv}\{f_1, \ldots, f_k\}, \operatorname{conv}\{f_{k+1}, \ldots\}) \geqq \epsilon \quad \text{if} \quad k \geqq 1.$$

The next two lemmas are needed to develop some characterizations of reflexivity that will be useful in establishing characterizations of super-reflexivity. It is known that every non-reflexive Banach space has an infinite-dimensional subspace with a non-shrinking basis and an infinite-dimensional subspace with a basis that is not boundedly complete [**5**, p. 374; **6**, p. 362]. We shall need quantitative measures of how "good" these bases can be, as described by means of the characteristic of the basis. This is given by Lemmas 1 and 2. The proofs of Lemmas 1 and 2 are similar to the argument on pages 116–117 of [**2**], but these lemmas give more information. In fact, Lemma 2 is a combination of (31) and (35) in [**2**].

It is known that a sequence $\{x_i\}$ in a Banach space is a basis for its closed linear span if and only if there is a positive number ϵ such that

$$\|\textstyle\sum_1^{n+p} a_i x_i\| \geqq \epsilon \|\sum_1^n a_i x_i\|$$

for all positive integers n and p and all numbers $\{a_i\}$. The largest such number ϵ is the *characteristic* of the basis.

The proofs of Lemmas 1 and 2 make repeated use of the following form of *Helly's condition*. "Given linear functionals $f_1, \ldots, f_n$ on a Banach space B and numbers $c_1, \ldots, c_n$ and M, the following two statements are equivalent.

(i) $|\sum_1^n a_i c_i| \leqq M\|\sum_1^n a_i f_i\|$ for all numbers $\{a_i\}$.

(ii) For every $\epsilon > 0$, there is an x in B such that $\|x\| < M + \epsilon$ and $f_i(x) = c_i$ if $1 \leqq i \leqq n$."

LEMMA 1. *Let B be a non-reflexive Banach space. If $0 < \theta < 1$ and $0 < \epsilon < 1$, then there are sequences $\{z_i\}$ and $\{g_i\}$ in the interiors of the unit balls of B and B^* such that*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j, \tag{2}$$

and, for all positive integers n and p and all numbers $\{a_i\}$,

$$\|\textstyle\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})\| \geqq \frac{1}{3}\epsilon\|\sum_1^n a_i z_i\|. \tag{3}$$

Proof. Let θ and ϵ satisfy $0 < \theta < 1$ and $0 < \epsilon < 1$. Let F be a member of B^{**} for which $\|F\| < 1$ and

$$\operatorname{dist}(F, B^c) > \max\{\theta, \epsilon^{\frac{1}{2}}\},$$

where B^c is the canonical image of B in B^{**}. We shall show that a sequence $\{(z_n, g_n, H_n)\}$ can be chosen inductively so that $z_n \in B$, $g_n \in B^*$, $\{H_n\}$ is an increasing sequence of finite sets of linear functionals with B as their domains, and:

(a) $||z_n|| < 1$, $||g_n|| < 1$;
(b) $F(g_n) = \theta$ for all n;
(c) $g_i(z_j) = \theta$ if $i \leqq j$ and $g_i(z_j) = 0$ if $i > j$;
(d) $||h|| < 3\epsilon^{-\frac{1}{2}}$ and $F(h) = h(z_i)$ if $h \in H_n$ and $i \geqq n$;
(e) if $z \in \text{lin}\{z_1, \dots, z_n\}$, then there is an h in H_n with $|h(z)| \geqq \epsilon^{\frac{1}{2}}||z||$.

Since $||F|| > \theta$, we can choose g_1 so that $||g_1|| < 1$ and $F(g_1) = \theta$. Then $||g_1|| > \theta$ and we can choose z_1 so that $g_1(z_1) = \theta$ and $||z_1|| < 1$. Let H_1 contain a single member chosen by the procedure described below for determining H_{p+1}. Suppose that (z_i, g_i, H_i) have been chosen to satisfy (a)-(e) when $i \leqq p$, where $p \geqq 1$. Then g_{p+1} must satisfy

$$||g_{p+1}|| < 1, \quad F(g_{p+1}) = \theta, \quad g_{p+1}(z_j) = z_j{}^e(g_{p+1}) = 0 \quad \text{if} \quad j \leqq p.$$

For the last two of these three conditions, Helly's condition (i) becomes

$$\theta \leqq M||\textstyle\sum_1^p a_i z_i{}^e + F|| \quad \text{for all } \{a_i\}.$$

Since this is satisfied if $M = \theta/\text{dist}(F, B^e) < 1$, g_{p+1} can be chosen to satisfy $||g_{p+1}|| < 1$. Now z_{p+1} must satisfy

$$||z_{p+1}|| < 1, \quad g_i(z_{p+1}) = \theta \quad \text{if} \quad i \leqq p+1, \quad h(z_{p+1}) = h(z_p) \quad \text{if} \quad h \in H_p.$$

For the last two of these three conditions, Helly's condition (i) becomes

$$|\theta \textstyle\sum_1^{p+1} a_i + h(z_p)| \leqq M||\textstyle\sum_1^{p+1} a_i g_i + h||$$

for all $\{a_i\}$ and all $h \in \text{lin}(H_p)$. Since

$$|\theta \textstyle\sum_1^{p+1} a_i + h(z_p)| = |F(\textstyle\sum_1^{p+1} a_i g_i + h)| \leqq ||F||\,||\textstyle\sum_1^{p+1} a_i g_i + h||$$

and $||F|| < 1$, we can let $M = ||F||$ and choose z_{p+1} so that $||z_{p+1}|| < 1$. Now let G_p be a finite set of linear functionals with unit norms and domains B which contains suitable linear functionals so that, for each z in $\text{lin}\{z_1, \dots, z_{p+1}\}$, there is a g in G_p with $|g(z)| \geqq \epsilon^{\frac{1}{2}}||z||$. Let us now show that, for each g in G_p, there is an h in B^* such that

$$||h|| < 3\epsilon^{-\frac{1}{2}}, \quad F(h) = g(z_{p+1}), \quad z_i{}^e(h) = z_i{}^e(g) \quad \text{if} \quad i \leqq p+1. \tag{4}$$

For the last two of these conditions, Helly's condition (i) becomes

$$|a \cdot g(z_{p+1}) + \textstyle\sum_1^{p+1} a_i z_i{}^e(g)| \leqq M||aF + \textstyle\sum_1^{p+1} a_i z_i{}^e|| \quad \text{for all } \{a_i\} \text{ and } a. \tag{5}$$

Since

$$\begin{aligned} |a \cdot g(z_{p+1}) + \textstyle\sum_1^{p+1} a_i z_i{}^e(g)| &= |g(az_{p+1} + \textstyle\sum_1^{p+1} a_i z_i)| \leqq ||az_{p+1} + \textstyle\sum_1^{p+1} a_i z_i|| \\ &\leqq ||aF + \textstyle\sum_1^{p+1} a_i z_i{}^e|| + ||aF - az^e{}_{p+1}|| \\ &\leqq (1 + [||F - z^e{}_{p+1}||/||F + \textstyle\sum_1^{p+1} a_i z_i{}^e/a||]) \\ &\qquad\qquad \times ||aF + \textstyle\sum_1^{p+1} a_i z_i{}^e|| \\ &\leqq (1 + 2\epsilon^{-\frac{1}{2}})||aF + \textstyle\sum_1^{p+1} a_i z_i{}^e||, \end{aligned}$$

we can satisfy (5) with $M = 1 + 2\epsilon^{-\frac{1}{2}}$ and choose h so that $\|h\| < 3\epsilon^{-\frac{1}{2}}$. It follows from (4) that $h \equiv g$ on $\text{lin}\{z_1, \ldots, z_{p+1}\}$. Let each member of G_p be replaced in this way and then let H_{p+1} be the union of H_p and all such replacements of members of G_p. Clearly the sequence $\{(z_i, g_i)\}$ satisfies (2). It follows from (e) that, for any sum $\sum_1^n a_i z_i$, there is an h in H_n such that

$$|h(\textstyle\sum_1^n a_i z_i)| \geqq \epsilon^{\frac{1}{2}} \|\textstyle\sum_1^n a_i z_i\|.$$

Since $\|h\| < 3\epsilon^{-\frac{1}{2}}$ and $h(z_i - z_{i-1}) = 0$ if $i > n$, we have

$$\begin{aligned} \|\textstyle\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})\| &\geqq \tfrac{1}{3}\epsilon^{\frac{1}{2}} |h[\textstyle\sum_1^n a_i z_i + \sum_{n+1}^{n+p} a_i(z_i - z_{i-1})]| \\ &= \tfrac{1}{3}\epsilon^{\frac{1}{2}} |h(\textstyle\sum_1^n a_i z_i)| \geqq \tfrac{1}{3}\epsilon \|\textstyle\sum_1^n a_i z_i\|. \end{aligned}$$

LEMMA 2. *Let B be a non-reflexive Banach space. If* $0 < \theta < 1$ *and* $0 < \epsilon < 1$, *then there are sequences* $\{z_i\}$ *and* $\{g_i\}$ *in the interiors of the unit balls of B and* B^* *such that*

$$g_1(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j,$$

and, for all positive integers n and p and all numbers $\{a_i\}$,

$$\|\textstyle\sum_1^{n+p} a_i z_i\| \geqq \tfrac{1}{2}\epsilon \|\textstyle\sum_1^n a_i z_i\|. \tag{6}$$

Proof. Let θ and ϵ satisfy $0 < \theta < 1$ and $0 < \epsilon < 1$. Let F be a member of B^{**} for which $\|F\| < 1$ and

$$\text{dist}(F, B^c) > \max\{\theta, \epsilon^{\frac{1}{2}}\},$$

where B^c is the canonical image of B in B^{**}. We shall show that a sequence $\{(z_n, g_n, H_n)\}$ can be chosen inductively so that $z_n \in B$, $g_n \in B^*$, $\{H_n\}$ is an increasing sequence of finite sets of linear functionals with B as their domains, and:

(a) $\|z_n\| < 1$, $\|g_n\| < 1$;
(b) $F(g_n) = \theta$ for all n;
(c) $g_i(z_j) = \theta$ if $i \leqq j$ and $g_i(z_j) = 0$ if $i > j$;
(d) $\|h\| < 2\epsilon^{-\frac{1}{2}}$ and $F(h) = h(z_i) = 0$ if $h \in H_n$ and $i > n$;
(e) If $z \in \text{lin}\{z_1, \ldots, z_n\}$, then there is an h in H_n with $|h(z)| \geqq \epsilon^{\frac{1}{2}}\|z\|$.

Assuming that (z_i, g_i, H_i) have been chosen to satisfy (a)-(e) for $i \leqq p$, the choice of g_{p+1} is made exactly as in the proof of Lemma 1. Then z_{p+1} must satisfy

$$\|z_{p+1}\| < 1, \quad g_i(z_{p+1}) = \theta \quad \text{if} \quad i \leqq p + 1, \quad h(z_{p+1}) = 0 \quad \text{if} \quad h \in H_p.$$

For the last two of these conditions, Helly's condition (i) becomes

$$|\theta \textstyle\sum_1^{p+1} a_i| \leqq M \|\textstyle\sum_1^{p+1} a_i g_i + h\|$$

for all $\{a_i\}$ and all $h \in \text{lin}(H_p)$. Since

$$|\theta \textstyle\sum_1^{p+1} a_i| = |F(\textstyle\sum_1^{p+1} a_i g_i + h)| \leqq \|F\| \, \|\textstyle\sum_1^{p+1} a_i g_i + h\|$$

and $||F|| < 1$, we can let $M = ||F||$ and choose z_{p+1} so that $||z_{p+1}|| < 1$. The remaining argument is similar to that for Lemma 1, with (4) replaced by

$$||h|| < 2\epsilon^{\frac{1}{2}}, \quad F(h) = 0, \quad z_i^c(h) = z_i^c(g) \quad \text{if} \quad i \leqq p+1,$$

and (5) replaced by

$$\left|\sum_1^{p+1} a_i z_i^c(g)\right| \leqq M\left|\left|F + \sum_1^{p+1} a_i z_i^c\right|\right|.$$

The coefficient $\frac{1}{2}$ in (6) is the best possible. To see this, suppose θ is a positive number and $\{x^n\}$ is a normalized basic sequence in c_0 for which there is a continuous linear functional g such that $g(x^n) \geqq \theta$ for every n. We shall show that $\operatorname{char}\{x^n\} \leqq \frac{1}{2}$. Let $\{y^n\}$ be a subsequence of $\{x^n\}$ for which

$$\lim_{n\to\infty} y^n(i) = \alpha_i$$

exists for each i. Then $|\alpha_i| \leqq 1$ for every i. Also $g(x^n) \geqq \theta$ for every n implies $\sup\{|\alpha_i|\} > 0$. For an arbitrary $\epsilon > 0$, let $\{z^n\}$ be a subsequence of $\{y^n\}$ such that, for every n,

$$|z^n(i) - \alpha_i| < \epsilon \quad \text{if} \quad i \leqq p(n) < p(n+1),$$

where $p(n)$ is an integer for which $|z^k(i)| < \epsilon$ if $k < n$ and $i \geqq p(n)$. Then, for every k and r,

$$\left|\left|\sum_{i=1}^k z^{r+i} - \omega\right|\right| < k\epsilon + 1,$$

where $\omega(i) = k\alpha_i$ if $1 \leqq i \leqq p(r+1)$, $\omega(i) = (k-j)\alpha(i)$ if $p(r+j) < i \leqq p(r+j+1)$, and $\omega(i) = 0$ if $i > p(r+k)$. Choose r such that $\sup\{|\alpha_i| : i \leqq p(r)\} > M - \epsilon$, where $M = \sup\{|\alpha_i|\}$. Then choose $s > k + r$. It follows that

$$\left|\left|\sum_{i=1}^k z^{r+i} - \tfrac{1}{2}\sum_{i=1}^k z^{s+i}\right|\right| < \tfrac{1}{2}kM + 2(k\epsilon + 1),$$

$$\left|\left|\sum_{i=1}^k z^{r+i}\right|\right| > k(M - \epsilon) - k\epsilon.$$

Thus, $\operatorname{char}\{x^n\} \leqq \operatorname{char}\{z^n\} < [\frac{1}{2}M + 2(\epsilon + 1/k)]/[M - 2\epsilon]$. Since k and ϵ were arbitrary, $\operatorname{char}\{x^n\} \leqq \frac{1}{2}$.

THEOREM 3. *Each of the following is a necessary and sufficient condition for a Banach space B to be non-reflexive. (Equivalent conditions are obtained if the introductory phrases for* (I), (II) *and* (III) *are replaced by "For some positive numbers θ and ϵ," or the introductory phrases for* (IV) *and* (V) *are replaced by "For some positive number θ".)*

(I) *For all θ and ϵ such that $0 < \theta < 1$ and $0 < \epsilon < 1$, there is a basic sequence $\{x_i\}$ in B such that $||x_i|| \geqq \theta$ for every i, $||\sum_1^k x_i|| < 1$ for every k, and* $\operatorname{char}\{e_i\} \geqq \frac{1}{3}\epsilon$.

(II) *For all θ and ϵ such that $0 < \theta < 1$ and $0 < \epsilon < 1$, there are sequences $\{z_n\}$ and $\{g_n\}$ in the unit balls of B and B^*, respectively, such that $\{z_i\}$ is a basic sequence with* $\operatorname{char}\{e_i\} \geqq \frac{1}{2}\epsilon$ *and*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j.$$

(III) *For all* θ *and* ϵ *such that* $0 < \theta < 1$ *and* $0 < \epsilon < 1$, *there is a basic sequence* $\{z_n\}$ *in the unit ball of* B *such that* $\operatorname{char}\{z_n\} \geqq \frac{1}{2}\epsilon$ *and*

$$\|z\| \geqq \theta \quad \text{if} \quad z \in \operatorname{conv}\{z_n\}.$$

(IV) *For all* θ *such that* $0 < \theta < 1$, *there is a sequence* $\{z_n\}$ *in the unit ball of* B *such that, for every sequence of numbers* $\{a_i\}$ *such that* $\sum_1^\infty a_i z_i$ *is convergent,*

$$\theta \cdot \sup\{|\textstyle\sum_k^\infty a_i| : k \leqq n\} \leqq \|\textstyle\sum_1^\infty a_i z_i\|. \tag{7}$$

(V) *For all* θ *such that* $0 < \theta < 1$, *there is a sequence* $\{x_n\}$ *in* B *such that, for every sequence of numbers* $\{a_i\}$ *for which* $\sum_1^\infty a_i x_i$ *is convergent and* $a_i \to 0$,

$$\theta \cdot \sup\{|a_i|\} \leqq \|\textstyle\sum_1^\infty a_i x_i\| \leqq \textstyle\sum_1^\infty |a_i - a_{i+1}|. \tag{8}$$

Proof. Suppose first that B is not reflexive. Let $\{(z_i, g_i)\}$ be as described in Lemma 1. Let $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ if $i > 1$. Then, for every i, $g_i(x_i) = \theta$ and therefore $\|x_i\| \geqq \theta$. Also, $\sum_1^k x_i = z_k$, so that $\|\sum_1^k x_i\| < 1$ for every k. Inequality (3) is equivalent to $\operatorname{char}\{x_i\} \geqq \frac{1}{3}\epsilon$. Thus (I) is satisfied. Clearly, (II) follows from Lemma 2 and (II) implies (III). Also, (II) implies (IV), since if $\{(z_i, g_i)\}$ are as described in (II), then

$$\theta \cdot \sup\{|\textstyle\sum_n^\infty a_i|\} = \sup\{|g_n(\textstyle\sum_1^\infty a_i z_i)|\} \leqq \|\textstyle\sum_1^\infty a_i z_i\|.$$

Let us now show that (IV) implies (V). To do this, let $\{z_n\}$ and θ be as described in (IV). Let $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ if $i > 1$. Then $\sum_1^\infty a_i x_i = \sum_1^\infty (a_i - a_{i+1}) z_i$, so that (7) and $\|z_i\| \leqq 1$ imply (8).

To complete the proof, it is sufficient to show that B is non-reflexive if (I), (III) or (V) is satisfied (note that the following arguments use only the existence of positive numbers θ and ϵ as described in (I)-(V), rather than the possibility of using arbitrary θ and ϵ in the interval (0,1)). If (I) or (III) is satisfied, then a subspace of B has a basis that is not boundedly complete or is not shrinking, so that B is not reflexive [**1**, Theorem 3, p. 71]. Now suppose θ and $\{x_n\}$ are as described in (V). For each n, let

$$K_n = \operatorname{cl}\{\textstyle\sum_1^p \alpha_i x_i : p \geqq n \quad \text{and} \quad 1 = \alpha_1 = \ldots = \alpha_n \geqq \alpha_{n+1} \geqq \ldots \geqq \alpha_p \geqq 0\}.$$

Then K_n is bounded, closed and convex, with $K_n \supset K_{n+1}$. Thus we can show B is non-reflexive by showing that $\bigcap K_n$ is empty [**1**, Theorem 1, p. 48]. Suppose $x \in \bigcap K_n$. Then there exist sequences $\{\alpha_i\}$ and $\{\beta_i\}$ that decrease monotonically to 0 for which

$$\|x - \textstyle\sum_1^p \alpha_i x_i\| < \frac{1}{2}\theta, \qquad \|x - \textstyle\sum_1^q \beta_i x_i\| < \frac{1}{2}\theta,$$

and $\beta_i = 1$ if $i \leqq p + 1$. Then $\|\sum_1^p \alpha_i x_i - \sum_1^q \beta_i x_i\| < \theta$, but from (8) we have

$$\|\textstyle\sum_1^p \alpha_i x_i - \textstyle\sum_1^q \beta_i x_i\| \geqq \theta\beta_{p+1} = \theta.$$

There are many properties of Banach spaces whose equivalence to non-super-reflexivity follows easily from the definition of super-reflexivity, but which will not be discussed in this paper (see Lemmas B and C and Theorem 6 of [**3**]). The first five characterizations in the next theorem are closely related to (I)-(V) of Theorem 3. Characterizations (vi) and (viii) are known [**4**, Theorem 6], but are included here to show their relation to (vii).

THEOREM 4. *Each of the following is a necessary and sufficient condition for a Banach space B not to be super-reflexive.* (*Equivalent conditions are obtained if the introductory phrases for* (*i*), (ii) *and* (iii) *are replaced by "For some positive numbers θ and ϵ," or the introductory phrases for* (iv) *and* (v) *are replaced by "For some positive number θ".*)

(i) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there is a subset $\{x_1, \dots, x_n\}$ of B such that $\|x_i\| \geqq \theta$ for every i, $\|\sum_1^k x_i\| < 1$ if $k \leqq n$, and, for every sequence of numbers $\{a_i\}$,*

$$\left\|\sum_1^n a_i x_i\right\| \geqq \tfrac{1}{3}\epsilon\left\|\sum_1^k a_i x_i\right\| \quad \text{if} \quad k \leqq n.$$

(ii) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there are subsets $\{z_1, \dots, z_n\}$ and $\{g_1, \dots, g_n\}$ of the unit balls of B and B^*, respectively, such that*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j, \quad g_i(z_j) = 0 \quad \text{if} \quad i > j,$$

and, for every sequence of numbers $\{a_i\}$ and every $k \leqq n$,

$$\left\|\sum_1^n a_i z_i\right\| \geqq \tfrac{1}{2}\epsilon\left\|\sum_1^k a_i z_i\right\|.$$

(iii) *If $0 < \theta < 1$ and $0 < \epsilon < 1$, then for every positive integer n there is a subset $\{z_1, \dots, z_n\}$ of the unit ball of B such that $\|z\| > \theta$ if $z \in \operatorname{conv}\{z_1, \dots, z_n\}$, and, for every sequence of numbers $\{a_i\}$ and every $k \leqq n$,*

$$\left\|\sum_1^n a_i z_i\right\| \geqq \tfrac{1}{2}\epsilon\left\|\sum_1^k a_i z_i\right\|.$$

(iv) *If $0 < \theta < 1$, then for every positive integer n there is a subset $\{y_1, \dots, y_n\}$ of the unit ball of B such that, for every sequence of numbers $\{a_i\}$,*

$$\theta \cdot \sup\left\{\left|\sum_k^n a_i\right| : k \leqq n\right\} \leqq \left\|\sum_1^n a_i y_i\right\|.$$

(v) *If $0 < \theta < 1$, then for every positive integer n there is a subset $\{x_1, \dots, x_n\}$ of B such that, for every sequence of numbers $\{a_i\}$ for which $a_{n+1} = 0$,*

$$\theta \cdot \sup\{|a_i| : 1 \leqq i \leqq n\} \leqq \left\|\sum_1^n a_i x_i\right\| \leqq \sum_1^n |a_i - a_{i+1}|.$$

(vi) *For every A, δ and B such that $0 < 2A < \delta \leqq 1 < B$, there exist numbers r and s for which $1 < r < \infty$, $1 < s < \infty$, and, if $\{e_i\}$ is any normalized basic sequence in B with $\operatorname{char}\{e_i\} \geqq \delta$, then*

$$A\left[\sum |a_i|^r\right]^{1/r} \leqq \left\|\sum a_i e_i\right\| \leqq B\left[\sum |a_i|^s\right]^{1/s},$$

for every sequence of numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

(vii) *There exist positive numbers δ, A and r such that $\delta < 1$, $1 < r < \infty$, and*

$$A\left[\sum |a_i|^r\right]^{1/r} \leqq \left\|\sum a_i e_i\right\|, \tag{9}$$

for every normalized basic sequence $\{e_i\}$ *with* $\operatorname{char}\{e_i\} \geqq \frac{1}{3}\delta$ *and every sequence of numbers* $\{a_i\}$ *such that* $\sum a_i e_i$ *is convergent.*

(viii) *There exist positive numbers* δ, B *and* s *such that* $\delta < 1$, $1 < s < \infty$, *and*

$$|\sum a_i e_i| \leqq B[\sum |a_i|^s]^{1/s}, \tag{10}$$

for every normalized basic sequence $\{e_i\}$ *with* $\operatorname{char}\{e_i\} \geqq \frac{1}{2}\delta$ *and every sequence of numbers* $\{a_i\}$ *such that* $\sum a_i e_i$ *is convergent.*

Proof. Observe first that if a Banach space B is not super-reflexive, then there is a non-reflexive space X that is finitely representable in B. The fact that X has each of properties (I)-(V) of Theorem 3 implies that B has each of properties (i)-(v). The proof that each of (i)-(v) implies there is a non-reflexive space X that is finitely representable in B is essentially the same as a known process that will not be repeated here (see the proof of Lemma B in [**3**]). This completes the proof of (i)-(v). It is known that (vi) is implied by super-reflexivity [**4**, Theorem 4]. Clearly (vi) implies both (vii) and (viii). Let us suppose that B is not super-reflexive, but that (vii) is satisfied. For δ, A and r as described in (vii), choose ϵ and n so that $\delta < \epsilon < 1$ and

$$n^{1/r}\delta A > 1.$$

For this ϵ and for $\theta = \delta$, choose $\{x_1, \ldots, x_n\}$ as described in (i). Since $\{x_1, \ldots, x_n\}$ can be extended to a basic sequence with characteristic greater than $\frac{1}{3}\delta$, (9) gives the contradiction:

$$n^{1/r}\delta A \leqq A[\textstyle\sum_1^n ||x_i||^r]^{1/r} \leqq ||\sum_1^n x_i|| < 1.$$

Similarly, if B is not super-reflexive, but (viii) is satisfied, choose ϵ and n so that $\delta < \epsilon < 1$ and

$$\theta n > Bn^{1/s}.$$

For this ϵ and for $\theta = \delta$, choose $\{z_1, \ldots, z_n\}$ as described in (iii). Since $\{z_1, \ldots, z_n\}$ can be extended to a basic sequence with characteristic greater than $\frac{1}{2}\delta$, (10) gives the contradiction

$$\theta n < ||\textstyle\sum_1^n z_1|| \leqq B[\sum ||z_i||^s]^{1/s} \leqq Bn^{1/s}.$$

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RESIDUE FREE DIFFERENTIALS AND THE CARTIER OPERATOR FOR ALGEBRAIC FUNCTION FIELDS OF ONE VARIABLE

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1. Introduction. Let K be a field of characteristic $p > 0$ and let A be a separably generated algebraic function field of one variable with K as its exact constant field. Throughout this paper we shall use the following notations to classify differentials of A/K:

$D(A)$: the K-module of all differentials,

$G(A)$: the K-module of all differentials of the first kind,

$R(A)$: the K-module of all residue free differentials in the sense of Chevalley [**2**, p. 48],

$E^*(A)$: the K-module of all pseudo-exact differentials in the sense of Lamprecht [**7**, p. 363], (compare the definition with our Lemma 8).

In the preceding paper [**5**, Satz 1] we proved that for a perfect field K,

$$\dim_K R(A)/E^*(A) = \dim_K G(A)/G(A) \cap E^*(A),$$
$$R(A) = G(A) + E^*(A).$$

This is a natural generalization of the theorem of Kunz [**6**, Satz 3], and this means that the K-dimension of $R(A)/E^*(A)$ is not greater than the genus $g(A/K)$.

In this paper we have two aims. One is to obtain similar formulae with the preceding formulae without the assumption of perfectness on K. Another is to consider a kind of transfer of these formulae on the constant field extension. The former consideration is necessary for the latter one.

In § 2 we shall study a definition and some fundamental properties of the Cartier operator which is an important tool for our aims. In particular we shall make comparison between a differential and the differential obtained by application of the operator on the given differential about their degrees and residues (Lemmas 2 and 3). In § 3 we shall introduce a semi-linear mapping $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ of $D(A)$ into $D(A_m)$. In relation with the notation A_m we explain some notations now. Let F and S be fields. Then F^{p^n} denotes the field of all elements of p^n-th power of elements in F for an integer n. $F\langle S\rangle$ denotes the extension field of F generated by adjunction of all elements of S to F. Following these notations A_m means $A\langle K_m\rangle$ with $K_m = K^{p^{-m}}$ $(m \geqq 0)$, and Cosp_{A/A_m} is the cotrace from A to A_m. By this mapping we shall define two K-submodules of $R(A)$

$$R_m(A) = \{\omega \in D(A) \mid C^m(\mathrm{Cosp}_{A/A_m}\omega) \in G(A_m)\}$$

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and

$$E_m(A) = \{\omega \in D(A) \mid C^m(\mathrm{Cosp}_{A/A_m}\omega) = 0\}.$$

After some discussions on residues and inseparable poles of differentials we shall characterize elements of $R(A)$ and $E^*(A)$ by the operator $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$, and we shall prove that $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$ (Corollary of Lemma 6), and determine an explicit form of a differential in $E^*(A)$ (Lemma 8), where c^* is the minimum integer of such n as $g(A\langle K_n\rangle/K_n) = g_0(A/K)$: the conservative genus of A/K. Further we shall show that linear independency of a finite number of differentials over K mod $E^*(A)$ is transferred into the field obtained by the constant field extension $A\langle L\rangle$ with $L \supset K$ (Lemma 9), for a preparation to the proofs of Theorems 1 and 2. In the last part of the section we shall prove Theorem 1. From this we obtain that the K-dimension of $R(A)/E^*(A)$ is not greater than the conservative genus $g_0(A/K)$.

In § 4 we shall prove Theorem 2 which shows the corresponding transferred formulae by the constant field extension with the formulae in Theorem 1.

As main results we have

THEOREM 1. *Using the same notations mentioned above it holds that*

$$\dim_K R(A)/E^*(A) = \dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m),$$
$$R(A) = R_m(A) + E^*(A),$$

for any integer $m \geqq c^*$.

THEOREM 2. *Using the same notations mentioned above it holds for a given extension L of K that*

$$R(A\langle L\rangle) = L\,\mathrm{Cosp}_{A/A\langle L\rangle} R_m(A) + E^*(A\langle L\rangle)$$

for any integer $m \geqq c^*$, *and*

$$\dim_L R(A\langle L\rangle)/E^*(A\langle L\rangle) = \dim_K R(A)/E^*(A).$$

2. The Cartier operator. We begin with the definition of the Cartier operator. Let us denote a separating variable of A over K by x and let us always use it in this sense. With this x, $\{1, x \ldots, x^{p-1}\}$ forms the so-called p-base of A/B with $B = A^p\langle K\rangle$ and represents any $\omega \in D(A)$ in a unique form $\omega = \sum_{i=0}^{p-1} a_i x^i dx$, with $a_i \in B$. If $a_{p-1}^{p^{-1}}$ is in A, define $C_A(\omega) = a_{p-1}^{p^{-1}}\,dx$; hence $C_A(\omega)$ is in $D(A)$. Then we say that $C_A(\omega)$ is defined in A and call C_A the Cartier operator for A. We shall simply write C instead of C_A when no confusion might arise.

Let L be an extension of K. Then the constant field extension $A\langle L\rangle$ of A by L over K is separably generated over L, as A is separably generated over K, and x remains as a separating variable of $A\langle L\rangle/L$. Hence $\{1, x, \ldots, x^{p-1}\}$ remains as a p-base of $A\langle L\rangle/A^p\langle L\rangle$.

The following fundamental properties of C_A are obvious by making use of facts in Eichler's book [**4**, pp. 160–167].

(1) Whether $C_A(\omega)$ is definable or not, $C_A(\omega)$ itself is independent of the choice of a separating variable.

(2) If $C_A(\omega_1)$ and $C_A(\omega_2)$ are defined for $\omega_1, \omega_2 \in D(A)$, then

$$C_A(y_1^p\omega_1 + y_2^p\omega_2) = y_1C_A(\omega_1) + y_2C_A(\omega_2) \text{ with } y_1, y_2 \in A.$$

(3) If $\omega \in D(A)$ is exact, $C_A(\omega)$ is defined and $C_A(\omega) = 0$. Conversely, if $C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega)$ is defined and is equal to 0 for an extension L of K, then ω is exact (cf. [**2**, p. 118] on the definition of Cosp).

(4) Let $\Omega = \mathrm{Cosp}_{A/A\langle K_m\rangle}\,\omega$ for $\omega \in D(A)$. Then all $C^i(\Omega)$ $(i = 1, \ldots, m)$ are defined in $A\langle K_m\rangle$.

LEMMA 1. *Let $C_A(\omega)$ be defined for a differential $\omega \in D(A)$. If L is an extension of K, then $C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega)$ is defined and*

$$C_{A\langle L\rangle}(\mathrm{Cosp}_{A/A\langle L\rangle}\omega) = \mathrm{Cosp}_{A/A\langle L\rangle}C_A(\omega).$$

Proof. The proof is obvious from the fact that $\mathrm{Cosp}_{A/A\langle L\rangle}dx$ is a non-zero exact differential of $A\langle L\rangle$ and $\mathrm{Cosp}_{A/A\langle L\rangle}ydx = y\,\mathrm{Cosp}_{A/A\langle L\rangle}dx$.

LEMMA 2. *Let $C(\omega)$ be defined for a differential $\omega \in D(A)$. Then*

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1)$$

holds for any place $\mathfrak{g}$ of A, where $\nu\mathfrak{g}$ is the valuation with respect to a place $\mathfrak{g}$, (cf. [**5**, Hilfssatz 2], when K is perfect).

Proof. If K is perfect, then $K = K^p$, $B = A^p$. Accordingly $C(\omega)$ is defined for any $\omega \in D(A)$. Any uniformizing variable t is a separating variable since the residue field of the place $\mathfrak{g}$ of the uniformizing variable t is separable over K. By property (1) we have

$$\nu_{\mathfrak{g}}(\omega) = \min_i\{\nu_{\mathfrak{g}}(a_it^idt)\} \leqq \nu_{\mathfrak{g}}(a_{p-1}t^{p-1}dt),$$
$$\nu_{\mathfrak{g}}(t^{-1}dt) = -1.$$

From this we get

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1).$$

From now on we consider a general field K. Let K^* be the smallest perfect field containing K in the algebraic closure of K and let $A^* = A\langle K^*\rangle$ be the constant field extension of A by K^*. Let $\mathfrak{p}^*$ be the place of A^* lying above a place $\mathfrak{g}$ of A and let $e_{\mathfrak{p}^*/\mathfrak{g}}$ be the ramification index of $\mathfrak{p}^*$ w.r.t. $\mathfrak{g}$. If we denote $\Omega = \mathrm{Cosp}_{A/A^*}\omega$, then

$$\nu_{\mathfrak{p}^*}(C(\Omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{p}^*}(\Omega) + 1).$$

On the other hand the divisor $\mathfrak{M} = (\mathrm{Con}_{A/A^*}\omega)\,(\Omega)^{-1}$ is integral as well-known and this is independent of ω. Since for any $\omega \in D(A)$, $\nu_{\mathfrak{p}^*}(\mathrm{Con}_{A/A^*}\omega) =$

$e_{\mathfrak{p}^*/\mathfrak{g}}\nu_{\mathfrak{g}}(\omega)$, therefore we have

$$\nu_{\mathfrak{g}}(C(\omega)) + 1 \geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1) + e_{\mathfrak{p}^*/f}^{-1}(1 - p^{-1})(\nu_{\mathfrak{p}^*}(\mathfrak{M}) + e_{\mathfrak{p}^*/\mathfrak{g}} - 1)$$
$$\geqq p^{-1}(\nu_{\mathfrak{g}}(\omega) + 1).$$

This completes our proof.

From Lemma 2 we obtain easily

COROLLARY. *Let* ω *be in* $D(A)$ *and let us assume that* $C(\omega), \ldots, C^m(\omega)$ *are all defined for a large enough integer* $m \geqq 0$. *Then there exists an integer* n^* *such that* $\nu_{\mathfrak{g}}(C^n(\omega)) \geqq -1$ *for any place* $\mathfrak{g}$ *and integer* n *with* $n^* \leqq n \leqq m$.

Proof. Using Lemma 2, $\nu_{\mathfrak{g}}(\omega) \geqq -1$ implies $\nu_{\mathfrak{g}}(C(\omega)) \geqq -1$. If $\nu_{\mathfrak{g}}(\omega) \leqq -2$, then $\nu_{\mathfrak{g}}(C(\omega)) > \nu_{\mathfrak{g}}(\omega)$ because $\nu_{\mathfrak{g}}(C(\omega)) - \nu_{\mathfrak{g}}(\omega) \geqq (1 + \nu_{\mathfrak{g}}(\omega))\ (p^{-1} - 1) > 0$. Hence $\nu_{\mathfrak{g}}(C^{n'}(\omega)) \geqq -1$ for $n' \geqq -(1 + \nu_{\mathfrak{g}}(\omega))$. Since the number of poles of ω is finite, let n^* be the maximum of such n''s. Then this n^* satisfies our Corollary.

LEMMA 3. *Let* $C(\omega)$ *be defined for a* $\omega \in D(A)$. *Then* $\mathrm{Res}_{\mathfrak{g}}\omega = (\mathrm{Res}_{\mathfrak{g}}C(\omega))^p$ *for any place* $\mathfrak{g}$ (cf. [**5**, Hilfssatz 1] when K is perfect).

Proof. Using the same notations as in the proof of Lemma 2 we have

$$(\mathrm{Res}_{\mathfrak{p}^*}C(\mathrm{Cosp}_{A/A^*}\omega))^p = \mathrm{Res}_{\mathfrak{p}^*}\mathrm{Cosp}_{A/A^*}\omega.$$

If the residue field of $\mathfrak{g}$ is identified with a subfield of the residue field of $\mathfrak{p}^*$, then

$$\mathrm{Res}_{\mathfrak{p}^*}\mathrm{Cosp}_{A/A^*}\omega = \mathrm{Res}_{\mathfrak{g}}\omega \text{ [\textbf{2}, Theorem 11, p. 119]}.$$

Hence we have by Lemma 1

$$(\mathrm{Res}_{\mathfrak{g}}C(\omega))^p = \mathrm{Res}_{\mathfrak{g}}\omega.$$

3. The Cartier operator and residue free differentials. Let us denote the fields $K^{p^{-m}}$, $A\langle K^{p^{-m}}\rangle$ and $B\langle K^{p^{-m}}\rangle$ by K_m, A_m and B_m for any integer $m \geqq 0$. If $\mathfrak{g}$ is a place of A/K whose residue field is separable over the constant field K, we call $\mathfrak{g}$ a separable place (or separable).

We obtain as a simple generalization of [**2**, Corollary 4, p. 123]:

LEMMA 4. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ *be a finite number of places of* A/K. *Then there exists a finite purely inseparable extension* L *of* K *such that the place* $\mathfrak{p}_i$ *of* $A\langle L\rangle/L$ *lying above* $\mathfrak{g}_i$ *is separable for any* $i = 1, \ldots, n$.

Proof. Using the same notation as in Lemma 2 we denote the place of A^* lying above $\mathfrak{g}_i$ by $\mathfrak{p}_i^*$. By [**2**, Corollary 4, p. 123] there exists an intermediary field M_i between K^* and K having the following properties: M_i is a finite purely inseparable extension of K; the place $\mathfrak{p}_i'$ of $A\langle M_i\rangle$ (which lies above $\mathfrak{g}_i$) lying below $\mathfrak{p}_i^*$ is separable, and $e_{\mathfrak{p}_i^*/\mathfrak{p}_i'} = 1$. If we form the composite field $L = \bigcup_{i=1}^n M_i$, then L is a finite purely inseparable extension over K. If $\mathfrak{p}_i$ is the place of $A\langle L\rangle$ lying below $\mathfrak{p}_i^*$, then $\mathfrak{p}_i$ is lying above $\mathfrak{p}_i'$ and is separable since $\mathfrak{p}_i^*$ is unramified with respect to $A\langle L\rangle$ [**2**, Corollary 3, p. 123].

COROLLARY. *Let* $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ *be a finite number of places of* A/K. *Then there exists an integer* $m_0 \geqq 0$ *such that for* $m \geqq m_0$ *the place* $\mathfrak{p}_i$ *of* A_m *lying above* $\mathfrak{g}_i$ *is separable for each* $i = 1, \ldots, n$.

Proof. In the proof of Lemma 4 L is contained in K^* and L is a finite extension of K. Hence there exists the minimum integer m_0 satisfying $K \subseteq L \subseteq K_{m_0}$. Since $\mathfrak{p}_i^*$ is unramified with respect to $A\langle L\rangle$, hence so to A_{m_0}, and therefore $\mathfrak{p}_i^*$ is unramified with respect to A_m for $m \geqq m_0$, so that the place $\mathfrak{p}_i$ of A_m lying below $\mathfrak{p}_i^*$ is separable.

LEMMA 5. *Let* ω *be in* $D(A)$. *Then there exists an integer* $m \geqq 0$ *such that* $\nu_{\mathfrak{p}}(C^m(\mathrm{Cosp}_{A/A_m}\omega)) \geqq -1$ *for all places* $\mathfrak{p}$ *of* A_m/K_m, *and any pole* $\mathfrak{p}$ *of* $\mathrm{Cosp}_{A/A_m}\omega$ *is separable.*

Proof. Using the same notation as in Lemma 4 there exist a finite number of places $\mathfrak{g}$ of A/K lying below all poles $\mathfrak{p}^*$ of $\mathrm{Cosp}_{A/A^*}\omega$. We apply the Corollary of Lemma 4 to these finite number of places. Denoting by r any integer which satisfies the Corollary, the place $\mathfrak{p}'$ of A_r lying below $\mathfrak{p}^*$ is separable. Let $q = -\min\{0, \min \nu_{\mathfrak{p}^*}(\mathrm{Cosp}_{A/A^*}\omega)\}$ for all places $\mathfrak{p}^*$ of A^*/K^*, and let $m = \max\{r, q-1\}$. If $\nu_{\mathfrak{p}}(\Omega) \geqq 0$ for a place of A_m, then $\nu_{\mathfrak{p}}(C^j(\Omega)) \geqq 0$ for $0 \leqq j \leqq m$ by Lemma 2, where $\Omega = \mathrm{Cosp}_{A/A_m}\omega$. If $\nu_{\mathfrak{p}}(\Omega) < 0$, then $\nu_{\mathfrak{p}}(C^j(\Omega)) \geqq -1$ for $m \geqq j \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$ by the Corollary of Lemma 2. The fact that $m \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$ is obvious; as from $e_{\mathfrak{p}^*/\mathfrak{p}} = 1$, it holds that

$$\nu_{\mathfrak{p}}(\Omega) = \nu_{\mathfrak{p}^*}(\mathrm{Con}_{A_m/A^*}\Omega) \geqq \nu_{\mathfrak{p}^*}(\mathrm{Cosp}_{A/A^*}\omega) \geqq -q,$$

hence $m \geqq -1 - \nu_{\mathfrak{p}}(\Omega)$. Since the place $\mathfrak{p}^*$ of A^* lying above any pole $\mathfrak{p}$ of Ω is among the poles of $\mathrm{Cosp}_{A/A^*}\omega$, $\mathfrak{p}^*$ is unramified with respect to A_m, and hence $\mathfrak{p}$ is separable. This completes our proof.

Let us define a set of differentials for any integer $m \geqq 0$:

$$R_m(A) = \{\omega \in D(A);\ C^m(\mathrm{Cosp}_{A/A_m}\omega) \in G(A_m)\}.$$

LEMMA 6. *The mapping* $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ *is a semi-linear homomorphism of the* K-*submodule* $R_m(A)$ *of* $R(A)$ *onto the* K_m-*module* $G(A_m)$.

Proof. Since Cosp_{A/A_m} is an A-linear mapping of $D(A)$ into $D(A_m)$, it is obvious from property (2) in § 2 that $R_m(A)$ is a K-module and $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ is a semi-linear homomorphism of $R_m(A)$ into $G(A_m)$. If $u\mathrm{Cosp}_{A/A_m}dx$ is an element of $G(A_m)$, then $\omega = u^{p^m}x^{p^m-1}dx$ is in $D(A)$ and $C^m(\mathrm{Cosp}_{A/A_m}\omega) = u\mathrm{Cosp}_{A/A_m}dx$. Thus the mapping is surjective. Let $\mathfrak{p}$ be the place of A_m lying above a place $\mathfrak{g}$ of A. If $u\,\mathrm{Cosp}_{A/A_m}dx = C^m(\mathrm{Cosp}_{A/A_m}\omega)$ for $\omega \in R_m(A)$, then by Lemma 3 we have for any place

$$\mathrm{Res}_{\mathfrak{g}}\,\omega = (\mathrm{Res}_{\mathfrak{p}}\,u\,\mathrm{Cosp}_{A/A_m}dx)^{p^m} = 0,$$

as $u\mathrm{Cosp}_{A/A_m}dx$ belongs to $G(A_m)$. Therefore $R_m(A)$ is a K-submodule of $R(A)$.

COROLLARY. $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$.

Proof. We proved in Lemma 6 that $R_m(A) \subseteq R(A)$ for every $m \geqq 0$. Conversely, let $\omega \in R(A)$. By making use of Lemmas 3 and 5 there exists an integer m such that $\nu_{\mathfrak{p}}(C^m(\mathrm{Cosp}_{A/A_m}\omega)) \geqq -1$, $\mathrm{Res}_{\mathfrak{p}}\, C^m(\mathrm{Cosp}_{A/A_m}\omega) = 0$ for all places $\mathfrak{p}$ of A_m/K_m and all poles of $C^m(\mathrm{Cosp}_{A/A_m}\omega)$ are separable. Therefore $C^m(\mathrm{Cosp}_{A/A_m}\omega)$ is in $G(A_m)$, hence $\omega \in R_m(A)$. From the Corollary of Lemma 4 we can take all $m \geqq c^*$, hence $R(A) = \bigcup_{m=0}^{\infty} R_m(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$.

LEMMA 7. *Let us assume that* $g(A_m/K_m) = g(A_n/K_n)$ *for some two integers* $n \geqq m \geqq 0$. *Then* $R_m(A) \subseteq R_n(A)$.

Proof. From the definition of $R_m(A)$, $C^m(\mathrm{Cosp}_{A/A_m}\omega)$ belongs to $G(A_m)$ for any $\omega \in R_m(A)$. Let $\mathfrak{p}'$ be any place of A_n/K_n and let $\mathfrak{p}$ be the place of A_m/K_m lying below $\mathfrak{p}'$. Then using the Corollary of Lemma 2

$$\begin{aligned}\nu_{\mathfrak{p}'}\,(C^n(\mathrm{Cosp}_{A/A_n}\omega)) &\geqq \nu_{\mathfrak{p}'}\,(C^m(\mathrm{Cosp}_{A/A_n}\omega))\\ &= e_{\mathfrak{p}'/\mathfrak{p}}\nu_{\mathfrak{p}}(C^m(\mathrm{Cosp}_{A/A_m}\omega))\\ &\geqq 0\end{aligned}$$

by the equality of two genera. Hence $C^n(\mathrm{Cosp}_{A/A_n}\omega)$ is contained in $G(A_n)$, which means that $\omega \in R_n(A)$.

From Lemma 7 it holds that

$$R_m(A) \subseteq R_n(A) \text{ if } n \geqq m \geqq c^*.$$

Let us denote the kernel of $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ by $E_m(A)$. Then $E_m(A)$ is contained in $E^*(A)$ in the sense of Lamprecht as shown by the following Lemma.

LEMMA 8. ω *is in* $E_m(A)$ *if and only if* $\omega = \sum_i \alpha_i x_i^{q_i-1}dx_i$, $q_i = p^{m_i}$, *where all integers* $m_i \leqq m-1$, *all* $\alpha_i \in K$ *and all* $x_i \in A$.

Proof. If $\omega = \sum_i \alpha_i x_i^{q_i-1}dx_i$, $q_i = p^{m_i}$, with $m_i \leqq m-1$, $\alpha_i \in K$ and $x_i \in A$, then $C^m(\mathrm{Cosp}_{A/A_m}\omega) = 0$. We shall prove the "only if" part by induction on m.

Let $C(\mathrm{Cosp}_{A/A_1}\omega) = 0$; then ω is exact by the property (3) in § 2. Assume that our lemma is true for all $E_n(A)$ with $n \leqq m-1$. If $\omega \in E_m(A)$, then

$$C^m(\mathrm{Cosp}_{A/A_m}\omega) = C(\mathrm{Cosp}_{A_{m-1}/A_m}C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}\omega)) = 0,$$

hence $C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}\omega) = dz$, $z \in A_{m-1}$. Since A_{m-1} is purely inseparable over A, z has no variable pole. Hence $z = \sum_i \alpha_i^{p^{-(m-1)}} x_i$, where $\alpha_i \in K$, $x_i \in A$, and $dz = \sum_i \alpha_i^{p^{-(m-1)}}\mathrm{Cosp}_{A/A_{m-1}}dx_i$. By this we have

$$C^{m-1}(\mathrm{Cosp}_{A/A_{m-1}}(\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1}dx_i)) = 0,$$

which means

$$\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1}dx_i \equiv 0 \bmod E_{m-1}(A).$$

Hence by the induction assumption

$$\omega - \textstyle\sum_i \alpha_i x_i^{p^{m-1}-1}dx_i = \sum_j \beta_j y_j^{q_j-1}dy_j,$$

where $m_j \leqq m-2$, $\beta_j \in K$, $y_j \in A$. This completes our proof.

Let L be an extension of K and let S be a K-submodule of $D(A)$. Then we define an L-module

$$L\ \mathrm{Cosp}_{A/A\langle L\rangle}S = \{\textstyle\sum_i \alpha_i \mathrm{Cosp}_{A/A\langle L\rangle}\omega_i \mid \alpha_i \in L,\ \omega_i \in S\}.$$

COROLLARY. *If L is algebraic over K, then*

$$E_m(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_m(A).$$

Proof. It is obvious that $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_m(A) \subseteq E_m(A\langle L\rangle)$. Conversely if $\Omega \in E_1(A\langle L\rangle)$, then $\Omega = dz$, $z \in A\langle L\rangle$. As L is algebraic over K, z is written in such form $z = \sum_i \alpha_i x_i$ with $\alpha_i \in L$, $x_i \in A$. Hence

$$\Omega = \textstyle\sum_i \alpha_i \mathrm{Cosp}_{A/A\langle L\rangle}dx_i \equiv 0 \bmod L\mathrm{Cosp}_{A/A\langle L\rangle}E_1(A).$$

Assume that $E_n(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_n(A)$ for $n \leqq m$.

Let $\Omega = z^{p^m-1}dz$ be a purely pseudo-exact differential of $E_{m+1}(A\langle L\rangle)$, then $C\Omega = z^{p^{m-1}-1}dz \in E_m(A\langle L\rangle)$. By the induction assumption

$$C\Omega = \textstyle\sum_i \beta_i x_i^{q_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i,\ q_i = p^{m_i},$$

where $m_i \leqq m-1$, $\beta_i \in L$, $x_i \in A$. From this we have

$$C(\Omega - \textstyle\sum_i \beta_i^p x_i^{r_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i) = 0,\ r_i = p^{m_i+1},$$

hence

$$\Omega - \textstyle\sum_i \beta_i^p x_i^{r_i-1}\mathrm{Cosp}_{A/A\langle L\rangle}dx_i \equiv 0 \bmod E_1(A\langle L\rangle).$$

Since $E_1(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_1(A)$ as proved already, Ω is contained in $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_{m+1}(A)$. Since any differential in $E_{m+1}(A\langle L\rangle)$ is written as the linear combination of purely pseudo-exact differentials with the coefficients in L, by Lemma 8 it is contained in $L\ \mathrm{Cosp}_{A/A\langle L\rangle}E_{m+1}(A)$.

If we denote $\bigcup_{m=0}^{\infty}E_m(A)$ by $E^*(A)$, then $E^*(A)$ is a K-submodule of $R(A)$. Since $E_m(A) \subseteq E_n(A)$ for $m \leqq n$, $E^*(A) = \bigcup_{m=c^*}^{\infty}E_m(A)$. Therefore $E^*(A\langle L\rangle) = L\ \mathrm{Cosp}_{A/A\langle L\rangle}E^*(A)$ if L is algebraic over K.

LEMMA 9. *Let L be an extension of K and let $A' = A\langle L\rangle$. If $\omega_1, \ldots, \omega_h$ of $D(A)$ are linearly independent over K* mod $E^*(A)$, *then* $\mathrm{Cosp}_{A/A'}\ \omega_1, \ldots, \mathrm{Cosp}_{A/A'}\ \omega_h$ *are linearly independent over L* mod $E^*(A')$.

Proof. We shall show that if $\mathrm{Cosp}_{A/A'}\omega_1, \ldots, \mathrm{Cosp}_{A/A'}\omega_h$ are linearly dependent over L mod $E_m(A')$, then $\omega_1, \ldots, \omega_h$ are linearly dependent over K mod $E_m(A)$. This is sufficient to our proof. We shall prove it by the induction on m.

The case $m = 1$ has been proven by Rosenlicht [**8**, Lemma 5]. Hence assume that our lemma is true for all m less than n. Let $\sum_i \alpha_i \mathrm{Cosp}_{A/A'}\omega_i \equiv 0$ mod $E_n(A')$ for L elements $\{\alpha_i\}$, not all of which are zero. Then using the Corollary of Lemma 8 we have

$$\textstyle\sum_i \alpha_i^{p-1}\mathrm{Cosp}_{A_1/A_1'}C(\mathrm{Cosp}_{A/A_1}\omega_i) \equiv 0 \bmod E_{n-1}(A_1').$$

By the induction assumption

$$\sum_i a_i^{p^{-1}} C(\mathrm{Cosp}_{A/A_1}\omega_i) \equiv 0 \bmod E_{n-1}(A_1)$$

for K elements $\{a_i\}$, not all of which are zero. Using the Corollary of Lemma 8 again and Lemma 8 we have

$$\sum_i a_i^{p^{-1}} C(\mathrm{Cosp}_{A/A_1}\omega_i) = \sum_i b_i^{p^{-1}} \mathrm{Cosp}_{A/A_1} u_i^{p^{n(i)-1}-1} du_i,$$

where $n(i) \leqq n-1$, $b_i \in K$ and $u_i \in A$. Hence

$$C(\mathrm{Cosp}_{A/A_1}(\sum_i a_i\omega_i - \sum_i b_i u_i^{p^{n(i)}-1} du_i)) = 0,$$

therefore we have $\sum_i a_i\omega_i \equiv 0 \bmod E_n(A)$. This completes our proof.

Corollary. *If* $g(A/K) = g(A'/L)$, *then it holds that*

$$\dim_K G(A)/G(A) \cap E^*(A) = \dim_L G(A')/G(A') \cap E^*(A').$$

In particular if $m, n \geqq c^*$, *then*

$$\dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m) = \dim_{K_n} G(A_n)/G(A_n) \cap E^*(A_n).$$

Proof. Let $\omega_1, \dots, \omega_h$ be a base of $G(A)$ mod $G(A) \cap E^*(A)$ over K. If $g(A/K) = g(A'/L)$, then $\mathrm{Cosp}_{A/A'}\omega_1, \dots, \mathrm{Cosp}_{A/A'}\omega_h$ are contained in $G(A')$ and by Lemma 9 they are linearly independent over L mod $G(A') \cap E^*(A')$. Hence we have

$$\dim_K G(A)/G(A) \cap E^*(A) \leqq \dim_L G(A')/G(A') \cap E^*(A').$$

Conversely it is obvious that a linear dependency of $\omega_1, \dots, \omega_h$ mod $E^*(A)$ over K implies a linear dependency of $\mathrm{Cosp}_{A/A'}\omega_1, \dots, \mathrm{Cosp}_{A/A'}\omega_h$ mod $E^*(A')$ over L. Hence the first equality holds. The second one follows easily from $g(A_m/K_m) = g(A_n/K_n)$.

The proof of Theorem 1. Let $R(A) = \bigcup_{m=c^*}^{\infty} R_m(A)$ and $E^*(A) = \bigcup_{m=c^*}^{\infty} E(A)$. Since $C^m(\mathrm{Cosp}_{A/A_m}\cdot)$ is the semi-linear homomorphism of $R_m(A)$ onto $G(A_m)$ whose kernel is $E_m(A)$, we have

$$R_m(A)/E_m(A) \cong G(A_m),\ \dim_K R_m(A)/E_m(A) = \dim_{K_m} G(A_m) = g_0(A/K).$$

At the same time, by this mapping $R_m(A) \cap E^*(A)$ is mapped onto $G(A_m) \cap E^*(A_m)$, hence by the Corollary of Lemma 9 we obtain

$$\dim_K R_m(A)/R_m(A) \cap E^*(A) = \dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m)$$

and this dimension is equal to $\dim_{K_{c^*}} G(A_{c^*})/G(A_{c^*}) \cap E^*(A_{c^*})$ for any $m \geqq c^*$. On the other hand we have

$$R(A)/E^*(A) = \bigcup_{m=c^*}^{\infty} (R_m(A) + E^*(A))/E^*(A),$$

$$\dim_K (R_m(A) + E^*(A))/E^*(A) = \dim_K R_m(A)/R_m(A) \cap E^*(A).$$

As for $n \geqq m$,

$$R_n(A) + E^*(A)/E^*(A) \supseteq R_m(A) + E^*(A)/E^*(A);$$

by Lemma 7, these two modules are identical and

$$R(A)/E^*(A) \cong R_m(A)/R_m(A) \cap E^*(A)$$

for any $m \geqq c^*$. Therefore we obtain the two equalities in Theorem 1.

COROLLARY. *If A/K is conservative, then*

$$\dim_K R(A)/E^*(A) = \dim_K G(A)/G(A) \cap E^*(A),$$
$$R(A) = G(A) + E^*(A).$$

(cf. [**5**, Satz 1] and [**6**, Satz 3]).

4. Residue free differentials and constant field extension. Let L be an extension of K and let $A' = A\langle L\rangle$.

LEMMA 10. *It holds that*

$$R_m(A') \subseteq L\,\mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

In particular if $g(A_m/K_m) = g(A'_m/L_m)$, then equality holds.

Proof. By definition, a differential $\Omega \in D(A')$ is in $R_m(A')$ if and only if $C^m(\mathrm{Cosp}_{A'/A_m'}\,\Omega)$ is in $G(A_m')$. Any differential in $G(A_m')$ is a linear combination of differentials in $\mathrm{Cosp}_{A_m/A_m'}G(A_m)$ with the coefficients in L_m. Hence $C^m(\mathrm{Cosp}_{A'/A_m'}\Omega)$ is written as follows:

$$C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) = \textstyle\sum_i \lambda_i^{p^{-m}} \mathrm{Cosp}_{A_m/A_m'}\Omega_i,$$

where $\Omega_i \in G(A_m)$, $\lambda_i \in L$. On the other hand $C^m(\mathrm{Cosp}_{A/A_m}\,\cdot)$ is the mapping of $R_m(A)$ onto $G(A_m)$, hence $\Omega_i = C^m(\mathrm{Cosp}_{A/A_m}\omega_i)$ with $\omega_i \in R_m(A)$. Therefore

$$C^m(\mathrm{Cosp}_{A'/A_m'}(\Omega - \textstyle\sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i)) = 0,$$

and hence $\Omega - \sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i \equiv 0 \mod E_m(A')$. This proves the first assertion.

Assume $g(A_m/K_m) = g(A_m'/L_m)$ and let $\Omega = \sum_i \lambda_i \mathrm{Cosp}_{A/A'}\omega_i$ be in $L\,\mathrm{Cosp}_{A/A'}R_m(A)$. Then

$$C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) = \textstyle\sum_i \lambda_i^{p^{-m}} \mathrm{Cosp}_{A_m/A_m'} C^m(\mathrm{Cosp}_{A/A_m}\omega_i)$$

with $C^m(\mathrm{Cosp}_{A/A_m}\omega_i) \in G(A_m)$. As $C^m(\mathrm{Cosp}_{A/A_m'}\omega_i)$ is in $G(A_m')$, $C^m(\mathrm{Cosp}_{A'/A_m'}\Omega) \in G(A_m')$, which means that Ω is in $R_m(A')$. Since $R_m(A') \supseteq E_m(A')$, we have

$$R_m(A') = L\,\mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

Proof of Theorem 2. By Theorem 1, $R(A') = R_m(A') + E^*(A')$ for $m \geqq c'^*$, where c'^* is in the same meaning with c^*. If $m \geqq c^*$, then $g(A_m'/L_m) = g(A_m/K_m)$ and by Lemma 10 we have

$$R_m(A') = L\,\mathrm{Cosp}_{A/A'}R_m(A) + E_m(A').$$

Since $R(A) = R_m(A) + E^*(A)$, it holds that

$$R(A') = L\,\mathrm{Cosp}_{A/A'} R_m(A) + E^*(A').$$

On the assertion of the dimensions, using the Corollary of Lemma 9 we have

$$\dim_{K_m} G(A_m)/G(A_m) \cap E^*(A_m) = \dim_{L_m} G(A_m')/G(A_m') \cap E^*(A_m')$$

for $m \geqq c^*$. Therefore by Theorem 1

$$\dim_K R(A)/E^*(A) = \dim_L R(A')/E^*(A').$$

This completes the proof of Theorem 2.

COROLLARY. $R(A') = L\,\mathrm{Cosp}_{A/A'} R(A) + E^*(A')$.

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THE HARDY SPACE H^1 ON MANIFOLDS AND SUBMANIFOLDS

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1. Introduction. It is well-known that the space $L^1(\mathbf{R}^n)$ of integrable functions on Euclidean space fails to be preserved by singular integral operators. As a result the rather large L^p theory of partial differential equations also fails for $p = 1$. Since L^1 is such a natural space, many substitute spaces have been considered. One of the most interesting of these is the space we will denote by $H^1(\mathbf{R}^n)$ of integrable functions whose Riesz transforms are integrable. Recall the Riesz transforms $R_1, \ldots, R_n$ are defined via the Fourier transform by

$$(R_j f)^\wedge(\xi) = \frac{\xi_j}{|\xi|}\hat{f}(\xi).$$

These are the n-dimensional analogues of the Hilbert transform (if $n = 1$ then R_1 is the Hilbert transform).

Now Stein [6] has shown that $H^1(\mathbf{R}^n)$ is preserved by all sufficiently smooth singular integral operators. In this paper we use that result to extend the basic L^p results used in the theory of elliptic boundary value problems to the class H^1. We will show that $H^1(\mathbf{R}^n)$ is locally preserved by pseudo-differential operators of order zero. This enables us to give an invariant definition of $H^1(M)$ where M is any compact C^∞ manifold without boundary: $H^1(M)$ is the space of all $f \in L^1(M)$ such that $Tf \in L^1(M)$ for every pseudo-differential operator T of order zero. We may also define Sobolev spaces $H_\alpha{}^1(M)$ of distributions having α derivatives in $H^1(M)$.

Next we characterize the restrictions of functions in $H_\alpha{}^1(M)$ to open submanifolds and lower dimensional submanifolds. After simple reductions of these problems to the Euclidean case our main results are as follows:

(1) A function $f(x, t) \in L^1(\mathbf{R}^{n-1} \times (0, \infty))$ is the restriction of an $H^1(\mathbf{R}^n)$ function if and only if the odd reflection

$$F(x, t) = \begin{cases} f(x, t) & \text{if } t > 0 \\ -f(x, -t) & \text{if } t < 0 \end{cases}$$

is in $H^1(\mathbf{R}^n)$.

(2) A function in $H_\alpha{}^1(\mathbf{R}^n)$ with $\alpha \geqq 1$ has a well-defined restriction to any hyperplane. Furthermore if $\alpha > 1$ the exact class of such restrictions is the Besov space $\Lambda(\alpha - 1; 1, 1)$. The case $\alpha = 1$ remains open.

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We will use freely without reference material found in Stein [**6**]. Most of our arguments are elementary, but they are based on two deep theorems of Stein and Fefferman (Theorems A and B below).

It is a great pleasure to acknowledge the assistance of my wife in proving Lemma 2.

2. The definition of $H^1(M)$. We consider $H^1(\mathbf{R}^n)$ as a Banach space with norm

$$\|f\|_{H^1} = \|f\|_1 + \sum_{j=1}^{n} \|R_j f\|_1.$$

We use the following result contained in Stein [**6**, p. 232]:

THEOREM A. *Let $m(\xi) \in L^\infty(\mathbf{R}^n)$ be C^{n+1} away from the origin and satisfy*

$$(1) \qquad \left| |\xi|^{|\alpha|} (\partial/\partial\xi)^\alpha m(\xi) \right| \leqq M \text{ for } |\alpha| \leqq n+1.$$

Then the multiplier transformation $Tf\,\hat{}\,(\xi) = m(\xi)\hat{f}(\xi)$ is bounded on $H^1(\mathbf{R}^n)$ with norm dominated by a multiple of M.

It is a simple matter to obtain from this an analogous local result for pseudo-differential operators of order zero.

THEOREM 1. *Let $p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ have compact support in the x-variable and satisfy*

$$(2) \quad |(\partial/\partial\xi)^\alpha (\partial/\partial x)^\beta p(x,\xi)| \leqq M_{\alpha,\beta}(1+|\xi|)^{-|\alpha|} \text{ for } |\alpha| \leqq n+1,\ |\beta| \leqq n.$$

Then the operator

$$Tf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)\,d\xi$$

is bounded from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$ with a norm depending only on the $M_{\alpha,\beta}$ and the support of $p(x, \xi)$.

Proof. We will show in fact that

$$\int \sup_y \left| \int e^{ix\cdot\xi} p(y,\xi)\hat{f}(\xi)\,d\xi \right| dx \leqq c\|f\|_H.$$

Indeed by Sobolev's inequality

$$\sup_y \left| \int e^{ix\cdot\xi} p(y,\xi)\hat{f}(\xi)\,d\xi \right| \leqq c \sum_{|\beta|\leqq n} \int \left| \int e^{ix\cdot\xi} (\partial/\partial y)^\beta p(y,\xi)\hat{f}(\xi)\,d\xi \right| dy.$$

On the other hand by Theorem A we have

$$\int \left| \int e^{ix\cdot\xi} (\partial/\partial y)^\beta p(y,\xi)\hat{f}(\xi)\,d\xi \right| dx$$
$$\leqq c \sup_\xi \sup_{|\alpha|\leqq n+1} |\xi|^{|\alpha|} |(\partial/\partial\xi)^\alpha (\partial/\partial y)^\beta p(y,\xi)| \, \|f\|_{H^1}.$$

Integrating with respect to y we obtain

$$\int \sup_y \left| \int e^{ix\cdot\xi} p(y, \xi) \hat{f}(\xi) d\xi \right| dx$$

$$\leqq c \int \sum_{|\beta| \leqq n} \sup_{|\alpha| \leqq n+1} \sup_{\xi} |\xi|^{|\alpha|} (\partial/\partial\xi)^{\alpha} (\partial/\partial y)^{\beta} p(y, \xi) | dy || f ||_{H^1}.$$

The integrand vanishes for y outside the support of $p(y, \xi)$ and is bounded because of (2), so the integral is finite.

At this point it is convenient to handle the problem of localizing H^1 functions. Note that multiplying by a function in $C^{\infty}{}_{\text{com}}(\mathbf{R}^n)$ will not preserve the class H^1 because all functions in H^1 must have total integral zero. The next lemma says in effect that this is the only difficulty.

LEMMA 1. *Let* $f(x) \in L^1(\mathbf{R}^n)$ *have compact support and total integral zero. If the Riesz transforms* $R_j f(x)$ *are integrable on a neighbourhood of the support of* f, *then* $f \in H^1(\mathbf{R}^n)$.

Proof. We must show that $R_j f(x)$ is integrable over the set of x whose distance to the support of f exceeds ϵ. Using the formula

$$R_j f(x) = c\text{P.V.} \int \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy$$

we see that $R_j f$ is bounded on this set. For large values of x we use the fact that $\int f(y) dy = 0$ to write

$$R_j f(x) = c \int \left(\frac{x_j - y_j}{|x - y|^{n+1}} - \frac{x_j}{|x|^{n+1}} \right) f(y) dy$$

and apply the mean value theorem to estimate

$$|R_j f(x)| \leqq c |x|^{-n-1} \int |y| \, |f(y)| dy$$

which is integrable.

We are now in a position to define $H^1(M)$ for a compact C^{∞} manifold without boundary M. We fix a smooth measure dx on M equivalent to Lebesgue measure in every coordinate system. Let $\{\varphi_i\}$ be a C^{∞} partition of unity subordinate to a covering by coordinate neighbourhoods and let ψ_i be a C^{∞} function supported in a coordinate neighbourhood satisfying $\psi_i \varphi_i = \varphi_i$. We define $H^1(M)$ to be the subspace of $L^1(M)$ of functions f for which $\psi_i R_j(\varphi_i f)$ is integrable for all i, j. Here the Riesz transform R_j is taken with respect to the local coordinate system. The norm on $H^1(M)$ is

$$|| f ||_1 + \sum_i \sum_j || \psi_i R_j(\varphi_i f) ||_1.$$

This definition appears to depend on the choice of the partition of unity and the local coordinates, but we shall see that in fact it does not.

Let us recall briefly the definition of pseudo-differential operators [**3**]. An operator $T\colon C^\infty(M) \to C^\infty(M)$ is called a pseudo-differential operator of order r if $\psi_i T(\varphi_i f)$ is given in local coordinates by

$$\int e^{ix\cdot\xi} p(x,\xi)(\varphi_i f)\hat{\ }(\xi)d\xi$$

where

$$p(x,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$$

satisfies

$$|(\partial/\partial\xi)^\alpha(\partial/\partial x)^\beta p(x,\xi)| \leqq M_{\alpha,\beta}(1+|\xi|)^{r-|\alpha|} \tag{3}$$

for all α and β, and $(1-\psi_i)T(\varphi_i f)$ is given by

$$\int_M K(x,y)f(y)dy$$

where $K \in C^\infty(M \times M)$.

THEOREM 1′. *If T is a pseudo-differential operator of order zero, then T is a bounded operator on $H^1(M)$.*

Proof. Since $f \to \psi_i R_j(\varphi_i f)$ is also a pseudo-differential operator of order zero, and these form an algebra under composition, it suffices to show $||Tf||_1 \leqq c||f||_{H^1}$. Now

$$Tf = \sum_i \psi_i T(\varphi_i f) + \sum_i (1-\psi_i)T(\varphi_i f),$$

so it suffices to estimate each term separately. We have easily

$$||(1-\psi_i)T(\varphi_i f)||_1 \leqq c||f||_1$$

so it remains to estimate $\psi_i T(\varphi_i f)$.

Now $\varphi_i f$ in local coordinates need not be in H^1, but this is easily remedied. We consider $\varphi_i f + g$ where g is chosen to make the total integral zero in local coordinates, and $g \in C^\infty_{\text{com}}$ with support on the set where $\psi_i = 1$. Easy estimates show $\psi_i R_j g \in L^1$, so by the hypotheses and Lemma 1 we have $\varphi_i f + g \in H^1(\mathbf{R}^n)$ in local coordinates (it is clear we may also choose g so that

$$||\varphi_i f + g||_{H^1} \leqq c\left(||f||_1 + \sum_{j=1}^n ||\psi_i R_j(\varphi_i f)||_1\right)\Bigg).$$

Applying Theorem 1 we obtain

$$||\psi_i T(\varphi_i f + g)||_1 \leqq c||\varphi_i f + g||_{H^1}.$$

But $||\psi_i Tg||_1$ can be easily estimated in terms of g and its derivatives, so we have

$$||\psi_i T(\varphi_i f)||_1 \leqq c\left(||f||_1 + \sum_{j=1}^n ||\psi_i R_j(\varphi_i f)||_1\right)$$

as desired.

COROLLARY. *A function f belongs to $H^1(M)$ if and only if $f \in L^1(M)$ and $Tf \in L^1(M)$ for every pseudo-differential operator T of order zero.*

Proof. If $f \in H^1(M)$ then $Tf \in L^1(M)$ by Theorem 1′. For the converse we have only to observe that $f \to \psi_i R_j(\varphi_i f)$ is a pseudo-differential operator of order zero.

We may now define the space $H_\alpha^1(M)$ for any real α to be the image of $H^1(M)$ under any invertible elliptic pseudo-differential operator of order $-\alpha$. Equivalently, $f \in H_\alpha^1(M)$ if and only if $Tf \in L^1(M)$ for every pseudo-differential operator T of order α.

3. Restrictions and extensions. Let us denote points in $\mathbf{R}^n$ by (x, t) with $x \in \mathbf{R}^{n-1}$ and $t \in R^1$. Let $g(x)$ be a real-valued function on R^{n-1} which is uniformly Lipschitz, $|g(x) - g(y)| \leqq A|x - y|$. Let $\Omega \subseteq \mathbf{R}^n$ be the set of points where $t > g(x)$. We define $H^1(\Omega)$ to be the set of restrictions to Ω of functions in $H^1(\mathbf{R}^n)$. We give $H^1(\Omega)$ the natural norm

$$\|f\|_{H^1} = \inf\{\|F\|_{H^1} : F \in H^1(\mathbf{R}^n) \quad \text{and} \quad F|\Omega = f\}.$$

Suitably interpreted, the above definitions make sense for $n = 1$ where $\Omega = \{t : t > g\}$ and g is a constant.

THEOREM 2. *Let $E : L^1(\Omega) \to L^1(\mathbf{R}^n)$ be the odd reflection*

$$Ef(x, t) = \begin{cases} f(x, t), & \text{if } t > g(x) \\ -f(x, 2g(x) - t), & \text{if } t < g(x) \end{cases}$$

(since the boundary $t = g(x)$ has measure zero we need not define Ef there). Then $f \in H^1(\Omega)$ if and only if $Ef \in H^1(\mathbf{R}^n)$, and $\|Ef\|_{H^1} \leqq c(A)\|f\|_{H^1}$.

To prove the theorem we make use of Fefferman's characterization of the dual space of $H^1(\mathbf{R}^n)$ [**2**]. We denote by BMO($\mathbf{R}^n$) the space of functions of bounded mean oscillation on $\mathbf{R}^n$ defined as follows: an element of BMO($\mathbf{R}^n$) is an equivalence class of locally integrable functions, modulo constant functions, satisfying

$$\frac{1}{m(Q)} \int_Q |f(x) - a_Q| dx \leqq M \tag{4}$$

for some constant a_Q and every cube $Q \subseteq \mathbf{R}^n$. The norm is the least such M. It is clear that the optimal value for a_Q is the mean value of f on the cube Q.

THEOREM B [**2**]. *The dual space of $H^1(\mathbf{R}^n)$ is* BMO($\mathbf{R}^n$) *under the usual pairing and with an equivalent norm.*

The restriction problem for BMO is easier to deal with than for H^1. Let us first observe that (4) holds for a larger class of sets. In fact we have

$$\frac{1}{m(D)} \int_D |f(x) - a_D| dx \leqq bM \tag{5}$$

for some constant a_D for every set D with the property that there exists a cube $Q \supseteq D$ such that $m(Q) \leqq bm(D)$. Indeed we need only set $a_D = a_Q$ and observe

$$\frac{1}{m(D)}\int_D |f(x) - a_Q|dx \leqq \frac{1}{m(D)}\int_Q |f(x) - a_Q|dx \leqq \frac{m(Q)}{m(D)} M.$$

We call such sets D b-quasicubes. We may then define $\mathrm{BMO}_b(\Omega)$ to be the set of equivalence classes of locally integrable functions on Ω for which (5) holds for every b-quasicube contained in Ω.

LEMMA 2. *There exists a constant depending only on n and A such that the following three conditions are equivalent* (*with norm equivalence*):

(i) $f \in \mathrm{BMO}_b(\Omega)$ for some $b \geqq c$;

(ii) there exists $F \in \mathrm{BMO}(\mathbf{R}^n)$ such that $F|\Omega = f$;

(iii) $E'f \in \mathrm{BMO}(\mathbf{R}^n)$, where E' is the even reflection

$$E'f(x, t) = \begin{cases} f(x, t), & \text{if } t > g(x) \\ f(x, 2g(x) - t), & \text{if } t < g(x). \end{cases}$$

Proof. Clearly (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) so it suffices to show (i) $\Rightarrow$ (iii). The gist of the argument is that because g satisfies a Lipschitz condition the reflection of a cube is a b-quasicube.

Let Q be any cube lying outside Ω, and let $D = \{(x, t) : (x, 2g(x) - t) \in Q\}$. We have $m(D) = m(Q)$, $D \subseteq \Omega$ and

$$\frac{1}{m(Q)}\int_Q |E'f(x, t) - a|dxdt = \frac{1}{m(D)}\int_D |f(x, t) - a|dxdt$$

so to establish (4) for Q it suffices to show that D is a b-quasicube. Now suppose Q is the cube $\{(x, t) : |x_j - y_j| \leqq r \text{ and } |t - s| \leqq r\}$. Then $m(Q) = (2r)^n$. Let $(x, 2g(x) - t) \in D$. Then $|x_j - y_j| \leqq r$ and $|2g(x) - t - (2g(y) - s)| \leqq r + 2|g(x) - g(y)| \leqq r + 2A\sqrt{n}r$ so D is contained in the cube with diameter $2r(1 + 2\sqrt{n}A)$ and centre $(y, 2g(y) - s)$. Thus D is a b-quasicube if $b \geqq (1 + 2\sqrt{n}A)^n$.

Finally we must verify (4) for cubes Q meeting the boundary of Ω. In this case we show by similar reasoning that the set D which is the union of $Q \cap \Omega$ with the reflection of $Q \cap -\Omega$ is a b-quasicube for sufficiently large b. We then have

$$\frac{1}{m(Q)}\int_Q |E'f(x, t) - a_D|dxdt \leqq \frac{2}{m(Q)}\int_D |f(x, t) - a_D|dxdt \leqq 2bM.$$

Proof of Theorem 2. Let $f \in H^1(\Omega)$ and let $F \in H^1(\mathbf{R}^n)$ with $F|\Omega = f$ and $||F||_{H^1} \leqq 2||f||_{H^1}$. First we claim that F may be approximated in $H^1(\mathbf{R}^n)$ norm by bounded functions of compact support. For if not there would exist a non-constant function $h \in \mathrm{BMO}(\mathbf{R}^n)$ such that $\int h(x)F(x)dx = 0$ for every $F \in H^1(\mathbf{R}^n)$ which is bounded with compact support. But Lemma 1 together with simple estimates shows that any bounded function with compact support and total integral zero is in $H^1(\mathbf{R}^n)$. Thus we may choose $F(x)$ to show that h is constant, a contradiction.

Let $\|F_k - F\|_{H^1} \leqq k^{-1}$ with F_k bounded with compact support. Let $f_k = F_k|\Omega$. Then $Ef_k \to Ef$ in L^1, so it suffices to show Ef_k is Cauchy in $H^1(\mathbf{R}^n)$ norm, and $\|Ef_k\|_{H^1} \leqq c\|F_k\|_{H^1}$.

Now observe that Ef_k is bounded with compact support and total integral zero, so $Ef_k \in H^1(\mathbf{R}^n)$. Thus

$$\|Ef_k\|_{H^1} \leqq \sup\left\{\int hEf_k : \|h\|_{\mathrm{BMO}} \leqq 1\right\}.$$

Now we write $h = h_1 + h_2$ where $h_1 = E'(h|\Omega)$. In view of Lemma 2 we have $\|h_1\|_{\mathrm{BMO}} \leqq c\|h\|_{\mathrm{BMO}}$ hence also $\|h_2\|_{\mathrm{BMO}} \leqq (c+1)\|h\|_{\mathrm{BMO}}$. But since h_1 is even and Ef_k is odd we have $\int h_1(x,t)Ef_k(x,t)dxdt = 0$. Thus

$$\int hEf_k = \int h_2 Ef_k = \int h_2 F_k$$

since h_2 vanishes off Ω. Thus $\|Ef_k\|_{H^1} \leqq c\|F_k\|_{H^1}$ and similarly

$$\|E(f_k - f_j)\|_{H^1} \leqq c\|F_k - F_j\|_{H^1}.$$

Remark. If $n = 1$ and $\Omega = \{t > 0\}$, the condition that $Ef \in H^1(\mathbf{R}^1)$ is equivalent to $f \in L^1(0, \infty)$ and

$$\text{P.V.} \int_0^\infty \frac{t}{x^2 - t^2} f(t)dt \in L^1(0,\infty).$$

For $n > 1$ there does not appear to be any simpler way to formulate the condition.

It is now a routine matter to generalize Theorem 2. If G is any open subset of M we may define $H^1(G)$ to be the space of restrictions to G of functions in $H^1(M)$.

COROLLARY 1. *If the boundary of G is a compact Lipschitz manifold then there exists a bounded linear extension map $E : H^1(G) \to H^1(M)$ so that $f \in H^1(G)$ if and only if $Ef \in H^1(M)$.*

We omit the proof and refer the reader to [**6**, Chapter 6] for similar arguments.

It is also possible to characterize restrictions to G of functions in $H_\alpha{}^1(M)$. For simplicity we consider only the case $\alpha = k$, a positive integer. Here we may define $H_k{}^1(G)$ to be the space of $f \in H^1(G)$ for which $Df \in H^1(G)$ for every differential operator D of order $\leqq k$.

COROLLARY 2. *There exists a bounded linear extension operator*

$$E_k : H_k{}^1(G) \to H_k{}^1(M).$$

Proof. By a partition of unity argument we may reduce the problem to finding $E_k : H_k^1(\Omega) \to H_k^1(\mathbf{R}^n)$, where $H_k^1(\Omega)$ is the space of $f \in H^1(\Omega)$ for which $D^\alpha f \in H^1(\Omega)$ for all α with $|\alpha| \leqq k$. We first extend f and $D^\alpha f$ to elements of $H^1(\mathbf{R}^n)$ by odd reflection, and then apply the construction of Calderón [**1**] for obtaining an extension operator from $L_k^p(\Omega) \to L_k^p(\mathbf{R}^n)$. We may apply Calderón's proof almost verbatim, using Theorem A for the preservation of $H^1(\mathbf{R}^n)$ by singular integrals.

Next we consider the problem of restrictions to hypersurfaces. Let us define $H_\alpha^1(\mathbf{R}^n)$ to be the image of $H^1(\mathbf{R}^n)$ under the Bessel potential G_α defined by $(G_\alpha f)\hat{}(\xi) = (1 + |\xi|^2)^{-\alpha/2}\hat{f}(\xi)$. Since G_α is an invertible elliptic pseudo-differential operator of order $-\alpha$ this is consistent with our previous definition of $H_\alpha^1(M)$, and it is not hard to see that $H_\alpha^1(M)$ is modelled on $H_\alpha^1(\mathbf{R}^n)$ in the same way that $H^1(M)$ is modelled on $H^1(\mathbf{R}^n)$.

There are many ways of defining the Besov spaces $\Lambda(\alpha; p, q)(\mathbf{R}^n)$. The following definition is due to Peetre [**4**]: let σ, τ be C^∞ functions on $[0, \infty)$ with $\sigma \equiv 1$ on $(1, 2)$ and supported on $(\frac{1}{2}, 4)$, and $\tau \equiv 1$ on $[0, 4)$ and supported on $[0, 8)$. Then $\Lambda(\alpha; p, q)(\mathbf{R}^n)$ is the space of tempered distributions on $\mathbf{R}^n$ for which the following norm is finite:

$$||f : \Lambda(\alpha; p, q)|| = ||\mathscr{F}^{-1}(\tau(|\xi|)\hat{f}(\xi))||_p + \left(\int_0^1 ||\mathscr{F}^{-1}(\sigma(t|\xi|)|\xi|^\alpha\hat{f}(\xi))||_p{}^q \frac{dt}{t}\right)^{1/q}$$

(if $q = \infty$, replace the integral by sup). It is clear from other equivalent definitions (see [**6**]) that $\Lambda(\alpha; p, q)$ is locally invariant under diffeomorphisms and multiplication by $C^\infty{}_{\text{com}}$ functions, so we may define $\Lambda(\alpha; p, q)(M)$ for a compact C^∞ manifold without boundary M in the usual manner.

THEOREM 3. *If $\alpha \geqq 1$, the restriction map $Rf(x, t) = f(x, 0)$ is well defined from $H_\alpha^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^{n-1})$. Furthermore, if $\alpha > 1$ then*

$$R : H_\alpha^1(\mathbf{R}^n) \to \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1})$$

and there exists an extension map $\mathscr{E} : \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \to H_\alpha^1(\mathbf{R}^n)$ such that $R\mathscr{E}f = f$.

Proof. If $f \in H_\alpha^1(\mathbf{R}^n)$ for $\alpha \geqq 1$ then $\partial f/\partial t \in L^1(\mathbf{R}^n)$ by Theorem A and

$$\int |f(x, 0)|dx \leqq \int_0^\infty \int_{\mathbf{R}^{n-1}} |(\partial f/\partial t)(x, t)|dxdt,$$

so $R : H_\alpha^1(\mathbf{R}^n) \to L^1(\mathbf{R}^{n-1})$. Now the boundedness of

$$R : H_\alpha^1(\mathbf{R}^n) \to \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \quad \text{for} \quad \alpha > 1$$

is essentially proved in Stein [**7**]. In fact, Stein proves the boundedness of $R : L_\alpha^p(\mathbf{R}^n) \to \Lambda(\alpha - 1/p; p, p)(\mathbf{R}^{n-1})$ if $\alpha > 1/p$ and $1 < p < \infty$ where $L_\alpha^p(\mathbf{R}^n)$ is the image of $L^p(\mathbf{R}^n)$ under G_α. But the same proof works for $p = 1$ (the restrictions $0 < \alpha < 1$ and $1/p < \alpha < 1$ on p. 579 of [**7**] may be replaced by $0 < \alpha < 2$ and $1/p < \alpha < 1 + 1/p$).

We define the extension map by means of the Fourier transform:

$$(6)\qquad (\mathscr{E}f)\hat{}(\xi,\eta) = c_1\tau(|\xi|)\hat{f}(\xi)\sigma(|\eta|) + c_2\frac{(1-\tau(|\xi|))|\xi|^{\beta-1}\hat{f}(\xi)}{(|\xi|^2+\eta^2)^{\beta/2}}$$

where $\beta > \alpha + 2 + n$ and c_1, c_2 are chosen so that

$$(7)\qquad \int_{-\infty}^{\infty} (\mathscr{E}f)\hat{}(\xi,\eta)\,d\eta = \hat{f}(\xi).$$

Now (7) means $R\mathscr{E}f = f$. To show $\mathscr{E} : \Lambda(\alpha - 1; 1, 1)(\mathbf{R}^{n-1}) \to H_\alpha^1(\mathbf{R}^n)$ we handle each term of (6) separately. The first term is trivial because it has compact support away from the origin. For the second term we use the identity

$$(1-\tau(|\xi|))\hat{f}(\xi) = \int_0^1 (1-\tau(|\xi|))\sigma^2(s|\xi|)\hat{f}(\xi)\frac{ds}{s}.$$

The problem is then to show that the following three expressions are Fourier transforms of L^1 functions:

$$(8)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha)/2}}\frac{ds}{s}$$

$$(9)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\xi_j\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}$$

$$(10)\qquad \int_0^1 \frac{|\xi|^{\beta-1}\eta\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}.$$

Let us consider for example (10) (the others are similar). We have

$$\left\| \mathscr{F}_{x,t}^{-1}\left(\int_0^1 \frac{|\xi|^{\beta-1}\eta\hat{f}(\xi)\sigma^2(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}\frac{ds}{s}\right)\right\|_1$$
$$\leqq \int_{-\infty}^{\infty}\int_0^1 \left\| \mathscr{F}_x^{-1}\left[\int_{-\infty}^{\infty}\frac{|\xi|^{\beta-\alpha}\eta\sigma(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}e^{it\eta}d\eta\right]\right\|_1$$
$$\times \|\mathscr{F}_x^{-1}(|\xi|^{\alpha-1}\sigma(s|\xi|)\hat{f}(\xi)\|_1 \cdot \frac{ds}{s}\,dt.$$

Since

$$\int_0^1 \|\mathscr{F}_x^{-1}(|\xi|^{\alpha-1}\sigma(s|\xi|)\hat{f}(\xi))\|_1 \frac{ds}{s} \leqq \|f : \Lambda(\alpha-1; 1, 1)\|$$

it suffices to show that

$$(11)\qquad \int_{-\infty}^{\infty}\left\| \mathscr{F}_x^{-1}\left(\int_{-\infty}^{\infty}\frac{|\xi|^{\beta-\alpha}\eta\sigma(s|\xi|)}{(|\xi|^2+\eta^2)^{(\beta-\alpha+1)/2}}e^{it\eta}d\eta\right)\right\|_1 dt$$

is bounded independent of s. But the change of variable $t \to st$, $\eta \to \eta/s$ and the dilation $\xi \to \xi/s$ (dilation on the Fourier transform side does not change the L^1 norm) shows that (11) is independent of s, so we may set $s = 1$. Next we make the change of variable $\eta \to |\xi|\eta$ to obtain

$$\int_{-\infty}^{\infty} \left\| \mathscr{F}_x^{-1} \left[\int_{-\infty}^{\infty} |\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta \right] \right\|_1 dt. \tag{12}$$

To estimate (12) we will use the well-known version of Sobolev's inequality,

$$\|\mathscr{F}_x^{-1}(g(\xi))\|_1 \leqq c \sum_{|\gamma| \leqq n+1} \|(\partial/\partial\xi)^\gamma g(\xi)\|_1. \tag{13}$$

Now

$$\int_{-\infty}^{\infty} \frac{\eta^k e^{ir\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta$$

is the kth derivative of the one-dimensional Bessel potential of order $\beta - \alpha + 1$. Thus we have

$$\left| \int_{-\infty}^{\infty} \frac{\eta^k e^{ir\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} d\eta \right| \leqq ce^{-Ar}$$

provided $\beta - \alpha + 1 - k > 1$ (see [**1**]). Thus

$$\left| \int_{-\infty}^{\infty} (\partial/\partial\xi)^\gamma \left(|\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+1)/2}} \right) d\eta \right| \leqq c\sigma(|\xi|)(1+|t|)^{|\gamma|} e^{-At|\xi|}$$

for $|\gamma| \leqq n + 1$, so by (13)

$$\left\| \mathscr{F}_x^{-1} \left(\int_{-\infty}^{\infty} |\xi|\sigma(|\xi|) \frac{\eta e^{it|\xi|\eta}}{(1+\eta^2)^{(\beta-\alpha+2)/2}} \right) d\eta \right\|_1 \leqq c(1+|t|)^{|\gamma|} e^{-At/2},$$

which shows (12) is bounded.

COROLLARY. *Let $N \subseteq M$ be a compact C^∞ submanifold of codimension one. Let $\partial/\partial t$ denote any transversal derivative to N. Let $T_k : C^\infty(M) \to C^\infty(N)^{k+1}$ be defined by $T_k f = (f|N, (\partial/\partial t)f|N, \ldots (\partial/\partial t)^k f|N)$. Then*

$$T_k : H_\alpha^1(M) \to \prod_{j=0}^{k} \Lambda(\alpha - 1 - j; 1, 1) \text{ for all } \alpha > k + 1,$$

and there exists an extension map

$$\mathscr{E}_k : \prod_{j=0}^{k} \Lambda(\alpha - 1 - j; 1, 1) \to H_\alpha^1(M)$$

such that $T_k \mathscr{E}_k$ is the identity.

We omit the routine proof of the above corollary, which is similar to one in Seeley [**5**].

Using the results of this section we can generalize to H^1 most of the results on elliptic boundary value problems valid for L^p, $1 < p < \infty$. We state one typical example from Seeley [**5**]:

THEOREM 4. *Let $M_0 \subseteq M$ be an open C^∞ submanifold with compact C^∞ boundary Γ, and let A be a uniformly elliptic differential operator on M of order m.*

Let u be any C^∞ function on M_0 which satisfies $Au = 0$ and which is the restriction to M_0 of a distribution on M. Then $u \in H_\alpha{}^1(M_0)$ if and only if the Cauchy data of u belongs to $\Pi_{j=0}^{m-1} \Lambda(\alpha - 1 - j; 1, 1)(\Gamma)$, for any non-negative integer α.

Remark. Comparing this with the results of [**8**], we see that for a solution of $Au = 0$ on M_0, $u \in H^1(M_0)$ if and only if $u \in L^1(M_0)$.

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COUNTEREXAMPLE TO A CONJECTURE ON POSITIVE DEFINITE FUNCTIONS

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1. Introduction. Cooper [**1**] called a complex-valued function f on the real line $\mathbf{R}$ *positive definite for* F, where F is a set of complex-valued functions on $\mathbf{R}$, if the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-y)\phi(x)\overline{\phi(y)}dxdy$$

exists as a Lebesgue integral and is nonnegative for every ϕ in F. Let us denote by $P(F)$ the set of functions positive definite for F, by L_c^p the set of functions in $L^p(\mathbf{R})$ with compact support, and by $L^p{}_{\text{loc}}$ the set of functions which are locally in $L^p(\mathbf{R})$, i.e., $f \in L^p{}_{\text{loc}} \Leftrightarrow f \in L^p(K)$ for every compact subset K of $\mathbf{R}$. Cooper showed that $P(L_c^p) = P(L_c^2)$ for any $p \geqq 2$ and that each function in $P(L_c^1)$ is essentially bounded and hence equal almost everywhere to an ordinary continuous positive definite function in the sense of Bochner. For $1 < p < 2$ he showed that if $q = p/2(p-1)$ then

$$P(L_c^2) \cap L^q{}_{\text{loc}} \subset P(L_c^p) \tag{1}$$

and he conjectured that equality holds in (1). In § 2 we give a counterexample to this conjecture, i.e., we construct a function $f \in P(L_c^p)$ which is not in $L^q{}_{\text{loc}}$.

The definition of $P(F)$ makes sense on any locally compact abelian group G and the inclusion (1) also holds in that situation (see [**4**, Theorem 2.3] and [**5**, Theorem 2.2]). If G is not a discrete group it seems unlikely that equality holds but the only group, apart from $\mathbf{R}$, for which we have discovered a counterexample is $G = T$, the circle group, and this appears in § 3.

In constructing the examples we shall need to use some of the theory of the Lorentz spaces $L(p, q)$, and we list here the relevant facts. If f is a measurable function defined on the measure space (X, μ) we define

$$m(f, y) = \mu\{x \in X : |f(x)| > y\} \text{ and } f^*(x) = \inf\{y > 0 : m(f, y) \leqq x\}.$$

For $1 \leqq p < \infty$, $1 \leqq q < \infty$, we let

$$L(p, q) = \left\{f : \int_0^{\infty} [x^{1/p} f^*(x)]^q x^{-1} dx < \infty\right\}.$$

It is well known that $L(p, p) = L^p$, that $L(p, q_1) \subset L(p, q_2)$ for $q_1 \leqq q_2$, and that

$$\int_X |f(x)g(x)| d\mu(x) \leqq \int_0^{\mu(X)} f^*(t)g^*(t)dt. \tag{2}$$

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It was proved by O'Neil [**3**, Theorem 2.6] that if $f \in L(p_1, q_1)$ and $g \in L(p_2, q_2)$, where $1/p_1 + 1/p_2 > 1$, then the convolution $f * g \in L(r, s)$ where $1/r = 1/p_1 + 1/p_2 - 1$ and $1/s \leqq 1/q_1 + 1/q_2$.

2. The counterexample for $G = \mathbf{R}$. Let $1 < p < 2$, $q = p/2(p-1)$, and $s = q^{-1}$. If $\rho(t) = t^s$ for $t \geqq 0$ and $\rho(t) = 0$ for $t < 0$, we define

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} e^{ixt} d\rho(t) = s \int_0^{\infty} e^{ixt} t^{s-1} dt \\ &= s\Gamma(s) e^{i\pi s/2} x^{-s} \qquad \text{if } x > 0. \end{aligned}$$

If $c = s\Gamma(s)e^{i\pi s/2}$ then

$$f(x) = \begin{cases} cx^{-s} & \text{if } x > 0 \\ \bar{c}(-x)^{-s} & \text{if } x < 0. \end{cases}$$

Clearly $f \notin L^q{}_{\text{loc}}$. We now show that the integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\phi(x)\overline{\phi(y)}dxdy = \int_{-\infty}^{\infty} f(x)\phi * \tilde{\phi}(x)dx \tag{3}$$

exists for all $\phi \in L_c^p(\tilde{\phi}(x) = \overline{\phi(-x)})$. Since ϕ and $\tilde{\phi}$ are in $L^p = L(p, p)$ we have, using the theorem of O'Neil,

$$g \equiv \phi * \tilde{\phi} \in L(r, 1) \quad \text{where} \quad r = p/(2-p) = q',$$

and so

$$\int_0^{\infty} x^{-s} g^*(x)dx = \int_0^{\infty} x^{1/q'-1} g^*(x)dx < \infty.$$

The decreasing rearrangement of f is easily calculated to be

$$f^*(x) = |c|(x/2)^{-s} = Kx^{-s}, \text{ say.}$$

Using (2) we have

$$\int_{-\infty}^{\infty} |f(x)g(x)|dx \leqq \int_0^{\infty} f^*(x)g^*(x)dx = K\int_0^{\infty} x^{-s}g^*(x)dx < \infty.$$

Thus the integral (3) exists.

In order to show that it is non-negative we can consider f as a distribution which is the Fourier transform of the distribution corresponding to the locally summable function $\Psi(t) = st^{s-1}(t > 0)$, $\Psi(t) = 0(t < 0)$. (See [**2**, § 2.3]). Thus we can write, for every $\phi \in C_c^{\infty}$,

$$\int f(x)\phi(x)dx = \int \hat{\phi}(t)d\rho(t)$$

where $\hat{\phi}$ is the Fourier transform of ϕ. Hence

$$\int f(x)\phi * \tilde{\phi}(x)dx = \int |\hat{\phi}(t)|^2 d\rho(t) \geqq 0 \qquad (\phi \in C_c^{\infty}).$$

If $\phi \in L_c^p$ we can find functions $\phi_n \in C_c^\infty$ such that $\phi_n(x) \to \phi(x)$ and $|\phi_n(x)| \leqq |\phi(x)|$. Now $|\phi_n * \tilde{\phi}_n| \leqq |\phi| * |\tilde{\phi}|$, and since $|\phi| \in L_c^p$ we have $\int f(x)(|\phi| * |\tilde{\phi}|)(x)dx < \infty$. Therefore by the dominated convergence theorem we have $\int f(x)\phi * \tilde{\phi}(x)dx \geqq 0$, i.e., $f \in P(L_c^p)$.

3. The counterexample for $G = \mathbf{T}$.

By analogy with the example of the previous section we define

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx}}{n^\alpha} \left(\alpha = 1 - \frac{1}{q}\right).$$

In view of the formula

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n^\alpha} = i\Gamma(1-\alpha)e^{-i\pi\alpha/2}x^{\alpha-1} + O(1) \quad (0 < x \leqq \pi)$$

(see [**6**, vol. **1**, p. **70**]) we can write

$$f(x) = \Gamma(q^{-1})e^{i\pi/2q}x^{-1/q} + h(x) = F(x) + h(x)$$

where $h \in L^\infty(T)$. Thus $f \notin L^q(T)$.

As in § 2 we let $\phi \in L^p(T)$ and then $g = \phi * \tilde{\phi} \in L(q', 1)$. Write

$$\int_0^{2\pi} |fg|dx \leqq \int_0^{2\pi} |Fg|dx + \int_0^{2\pi} |hg|dx.$$

Since $h \in L^\infty(T)$ and $g \in L^1(T)$ the second integral exists, whereas

$$\int_0^{2\pi} |Fg|dx \leqq \int_0^{2\pi} F^*(x)g^*(x)dx = K\int_0^{2\pi} x^{-1/q}g^*(x)dx < \infty.$$

Thus the integral

$$\int_0^{2\pi} f(x)\phi*\tilde{\phi}(x)dx$$

exists.

To show that it is nonnegative we consider the functions

$$f_N(x) = \sum_{n=1}^{N} \frac{e^{inx}}{n^\alpha}.$$

Since this is a finite trigonometric series with positive coefficients, each f_N is an ordinary positive definite function, and therefore

$$\int_0^{2\pi} f_N(x)\phi*\tilde{\phi}(x)dx \geqq 0 \text{ for every } \phi \in L^p.$$

We have $f_N(x) \to f(x)$ and also

$$|f_N(x)| \leqq C_\alpha x^{\alpha-1}$$

where C_α is a constant which depends on α but not on N (see [**6**, vol. I, p. **191**]).

Since $\int C_\alpha x^{\alpha-1}\phi * \tilde{\phi}(x)dx < \infty$, we can use the dominated convergence theorem to show that

$$\int_0^{2\pi} f(x)\phi * \tilde{\phi}(x)dx \geqq 0.$$

We have thus shown that $f \in P(L^p)$.

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REARRANGEMENT INEQUALITIES

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1. Introduction. In recent years a number of inequalities have appeared which involve rearrangements of vectors in $\mathbf{R}^n$ and of measurable functions on a finite measure space. These inequalities are not only interesting in themselves, but also are important in investigations involving rearrangement invariant Banach function spaces and interpolation theorems for these spaces [**2**; **8**; **9**].

The most famous inequality of this type for vectors is due to Hardy-Littlewood and Polya [**4**, Theorem 368]:

$$\sum_{i=1}^{m} a_i^* b_i' \leqq \sum_{i=1}^{m} a_i b_i \leqq \sum_{i=1}^{m} a_i^* b_i^* \tag{1.1}$$

with equality on the left (right) if and only if $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ are oppositely (similarly) ordered. Here the $a_i^*(a_i')$ are the numbers a_i in decreasing (increasing) order.

An example involving more than two vectors is the following one of H. D. Ruderman [**12**]:

$$\prod_{j=1}^{m} \sum_{k=1}^{n} a_{k,j} \geqq \prod_{j=1}^{m} \sum_{k=1}^{n} a_{k,j}^* \tag{1.2}$$

where $a_{k,j} > 0$ for all k, j, and for each k the $a_{k,j}^*$ are the numbers $a_{k,1}, \ldots, a_{k,m}$ in decreasing order. A condition for equality was not given.

Other inequalities of these types are possible, and general theorems have been given by G. G. Lorentz [**7**] and D. London [**6**].

Workers with inequalities generally recognize that many inequalities which are proved for real numbers by real variable methods also hold in more general systems. In Section 3 we let $\varphi : T_1 \times T_2 \rightarrow G$ where T_1, T_2 are ordered sets, and G is a partially ordered abelian group, and we give a necessary and sufficient condition on φ so that

$$\sum_{j=1}^{n} \varphi(a_j^*, b_j') \leqq \sum_{j=1}^{n} \varphi(a_j, b_j) \leqq \sum_{j=1}^{n} \varphi(a_j^*, b_j^*)$$

for all chains $\mathbf{a} \in T_1^n$, $\mathbf{b} \in T_2^n$. Also we give a necessary and sufficient condition on φ so that equality holds on the right (left) if and only if $\mathbf{a}$ and $\mathbf{b}$ are similarly (oppositely) ordered. We give a sufficient condition so that $\varphi(\mathbf{a}^*, \mathbf{b}') \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}^*)$, where $\ll$ denotes a preorder relation of

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Hardy, Littlewood, and Polya. Similar results to these are given when φ is a function of n variables.

W. A. J. Luxemburg [**9**] has proved analogs of discrete rearrangement inequalities for measurable functions on a finite measure space. In Section 5, all our discrete results are generalized for real valued essentially bounded measurable functions on a finite measure space. For specific choices of φ the inequalities are shown to hold for even larger classes of functions. The concept of "similarly ordered" is generalized for measurable functions to give a necessary and sufficient condition for equality.

Finally in Sections 4 and 6 we give numerous examples to show how to obtain many known rearrangement inequalities. Our analysis gives conditions for equality, in many cases for the first time.

2. Definitions and notation. Let T be a partially ordered set. If $\mathbf{a} = (a_1, \dots, a_m) \in T^m$, then $\mathbf{a}$ will be called a *chain* if $\{a_1, \dots, a_m\}$ is linearly ordered. If $\mathbf{a}$ is a chain, then $\mathbf{a}^* = (a_1^*, \dots, a_m^*)(\mathbf{a}' = (a_1', \dots, a_m'))$ denotes the vector obtained by rearranging the components of $\mathbf{a}$ in decreasing (increasing) order. If $\mathbf{a}$ and $\mathbf{b}$ are chains in a partially ordered abelian group G (written additively), then $\mathbf{b} \ll \mathbf{a}$ means $\sum_{i=1}^{k} b_i^* \leqq \sum_{i=1}^{k} a_i^*$ for all $1 \leqq k \leqq m$; and $\mathbf{b} \prec \mathbf{a}$ means $\mathbf{b} \ll \mathbf{a}$ and $\sum_{i=1}^{m} b_i^* = \sum_{i=1}^{m} a_i^*$. It will be notationally simpler and should cause no confusion to denote every partial order under consideration by $\leqq$. A partial order is understood to be anti-symmetric, and $x < y$ is used to mean $x \leqq y$ and $x \neq y$.

Let T_1 and T_2 be partially ordered sets. Chains $\mathbf{a} \in T_1^m$ and $\mathbf{b} \in T_2^m$ are said to be *similarly* (*oppositely*) *ordered* if for every $1 \leqq i, j \leqq m$, $a_i < a_j$ implies $b_i \leqq b_j$ $(b_j \leqq b_i)$.

Let $T_1, \dots, T_n$ be partially ordered sets, and let

$$\mathbf{a}_k = (a_{k,1}, \dots, a_{k,m}) \in (T_k)^m.$$

It is sometimes necessary to substitute values for some of the variables x_i in $(x_1, \dots, x_n)$ and then consider the result as a function of the remaining x_i. Let I, J, and K be disjoint subsets of $N = \{1, \dots, n\}$. To denote the result of substituting $a_{i,j}$ for x_i when $i \in I$, $a_{i,k}$ for x_i when $i \in J$, and $a_{i,l}$ for x_i when $i \in K$, we use the notation $(a_{I,j}, a_{J,k}, a_{K,l})$. In addition, $(\mathbf{a}_1, \dots, \mathbf{a}_n)$ denotes the sequence of vectors given by $j \mapsto (a_{1,j}, \dots, a_{n,j})$, and similarly for $(\mathbf{a}_I^*, \mathbf{a}_J')$ when $\{I, J\}$ is a partition of $\{1, \dots, n\}$.

Let $\varphi : T_1 \times \dots \times T_n \to G$. When I and J are partitions of $N = \{1, \dots, n\}$, we define conditions (A) and (A^*) on φ as follows.

(A) $\qquad$ $[(A^*)]$ If $x_i, y_i \in T_i$ with $x_i < y_i$, and $k \neq i$,

then $\varphi(y_i) - \varphi(x_i)$ is [strictly] increasing in u_k when k and i are in the same set I or J, and [strictly] decreasing in u_k when k and i are in different sets I and J, for all $1 \leqq i, k \leqq n$.

If $G = \mathbf{R}$, if $T_k = [r_k, s_k]$ with $r_k < s_k$, if the first partials of φ are separately continuous on $T_1 \times \ldots \times T_n$, and if the second partials of φ exist on $T =]r_1, s_1[\times \ldots \times]r_n, s_n[$, then [**1**, Theorems 5–7 and 5–10] implies that condition (A) is equivalent to:

$$(A)' \quad \partial^2\varphi/\partial u_i \partial u_j \begin{array}{l} \geqq 0, \text{ when } i \text{ and } j \text{ are in the same set } I \text{ or } J \\ \leqq 0, \text{ when } i \text{ and } j \text{ are in different sets } I \text{ and } J \end{array}$$

on T for all $1 \leqq i \neq j \leqq n$.

A sufficient differentiability condition for (A^*) is $(A^*)'$:

φ satisfies $(A)'$ and in addition, $\{u_i \in]r_i, s_i[: \partial^2\varphi/\partial u_i \partial u_j = 0\}$ contains no open interval for all $r_k < u_k < s_k$, and $1 \leqq k \neq i \leqq n$.

Let (X, Λ, μ) be a finite measure space with $\alpha = \mu(X) < \infty$, and let $M = M(X, \mu)$ denote the set of all extended real valued measurable functions on X. If $f \in M$, then the *decreasing rearrangement* δ_f of f is defined by

$$\delta_f(t) = \inf\{s \in \mathbf{R} : \mu(\{x : f(x) > s\}) \leqq t\} \text{ for } 0 \leqq t \leqq \alpha.$$

Also $\iota_f(t) = \delta_f((\alpha - t)-)$ denotes the *increasing rearrangement* of f, 1_E denotes the *characteristic function* of $E \in \Lambda$; $f|E$ denotes the *restriction of f to E*; and we let $I_f = [\text{ess. inf} f, \text{ess. sup} f]$.

If $f, g \in M$ then $f \sim g$ means $\delta_f = \delta_g$. This is equivalent to having $\mu(\{f > t\}) = \mu(\{g > t\})$ for all $t \in \mathbf{R}$. Let $(t_1, \ldots, t_n) > (u_1, \ldots, u_n)$ mean $t_i > u_i$, $1 \leqq i \leqq n$. For measurable $\mathbf{f}, \mathbf{g} : X \to \mathbf{R}^n$ we define $\mathbf{f} \sim \mathbf{g}$ to mean $\mu(\{\mathbf{f} > \mathbf{t}\}) = \mu(\{\mathbf{g} > \mathbf{t}\})$ for all $\mathbf{t} \in \mathbf{R}^n$.

We will say that $f, g \in M$ are *similarly ordered* if $\text{ess. sup} f|A < \text{ess. inf} f|B$ implies $\text{ess. sup}\, g|A \leqq \text{ess. inf}\, g|B$ whenever $A, B \in \Lambda$ are disjoint and each has positive measure. Analogously, $f, g \in M$ are called *oppositely ordered* if f and $-g$ are similarly ordered.

3. The discrete case.

This section is devoted to the proof of the following theorem.

(3.1) THEOREM. *Let $\varphi : T_1 \times \ldots \times T_n \to G$, where each T_k $(k = 1, \ldots, n)$ is linearly ordered, and G is a partially ordered abelian group. Let $\{I, J\}$ be a partition of $N = \{1, \ldots, n\}$.*

(i) *φ satisfies condition (A) if and only if*

$$\sum_{j=1}^{m} \varphi(a_{1,j}, \ldots, a_{n,j}) \leqq \sum_{j=1}^{m} \varphi(a_{I,j}{}^*, a_{J,j}{}') \tag{1}$$

for all $\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,m}) \in (T_k)^m$, $k = 1, \ldots, n$.

(ii) *φ satisfies condition (A^*) if and only if the following are equivalent for all $\mathbf{a}_k \in (T_k)^m$, $k = 1, \ldots, n$.*

(a) *Equality occurs in* (1).

(b) *$\mathbf{a}_p$ and $\mathbf{a}_q$ are similarly ordered whenever p and q are in the same set*

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I or J, and oppositely ordered when p and q are in different sets I and J, for all $1 \leqq p, q \leqq n$.

(c) $\varphi(\mathbf{a}_1, \dots, \mathbf{a}_n) \sim \varphi(\mathbf{a}_I^*, \mathbf{a}_J')$.

(iii) *Suppose the range of φ is linearly ordered. If φ satisfies condition (A) and is increasing (respectively decreasing) in u_k for $k \in I$ and decreasing (respectively increasing) in u_k for $k \in J$, then*

$$\varphi(\mathbf{a}_1, \dots, \mathbf{a}_n) \ll \varphi(\mathbf{a}_I^*, \mathbf{a}_J') \tag{2}$$

for all chains $\mathbf{a}_k \in T_k^m$ $(k = 1, \dots, n)$.

Proof. To prove necessity of (A) for (1), let $1 \leqq k, i \leqq n$, let $x_i, y_i \in T_i$ with $x_i < y_i$, let $\mathbf{a}_i = (x_i, y_i, \dots, y_i)$, let $u_k, v_k \in T_k$ with $u_k < v_k$, and for $j \neq i, k$ let $u_j \in T_j$ and $\mathbf{a}_j = (u_j, \dots, u_j)$. Case 1: k, i are in the same set I or J. Let $\mathbf{a}_k = (v_k, u_k, \dots, u_k)$. After cancelling terms in (1) we obtain

$$\varphi(x_i, v_k) + \varphi(y_i, u_k) \leqq \varphi(y_i, v_k) + \varphi(x_i, u_k),$$

so

$$\varphi(y_i, u_k) - \varphi(x_i, u_k) \leqq \varphi(y_i, v_k) - \varphi(x_i, v_k),$$

and hence (A) is true in this case. Case 2: k, i are in different sets I and J. Let $\mathbf{a}_k = (u_k, v_k, \dots, v_k)$. The proof is similar to Case 1. This completes the proof of necessity.

Before continuing we introduce some notation. For $\mathbf{a}_k \in T_k^m$ write $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ if $1 \leqq i < j \leqq m$ are such that for $P = \{k \in I : a_{k,i} < a_{k,j}\}$, and $Q = \{k \in J : a_{k,i} > a_{k,j}\}$ we have: $\mathbf{b}_k$ for $k \in P \cup Q$ is the sequence obtained from $\mathbf{a}_k$ by interchanging $a_{k,i}$ and $a_{k,j}$, while $\mathbf{b}_k = \mathbf{a}_k$ for other k.

Assume $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ with P and Q as above, and let $\psi = \varphi(a_{P,i}, a_{Q,i}) - \varphi(a_{P,j}, a_{Q,j})$. Also, for $0 \leqq k \leqq n$ let

$$P_k = P \cap \{0, \dots, k\} \quad \text{and} \quad Q_k = Q \cap \{0, \dots, k\}.$$

Then

$$\begin{aligned} \psi = {} & \sum_{k=0}^{n-1} [\varphi(a_{P,i}, a_{Q-Q_k,i}, a_{Q_k,j}) - \varphi(a_{P,i}, a_{Q-Q_{k+1},i}, a_{Q_{k+1},j})] \\ & + \sum_{k=0}^{n-1} [\varphi(a_{P-P_k,i}, a_{P_k,j}, a_{Q,j}) - \varphi(a_{P-P_{k+1},i}, a_{P_{k+1},j}, a_{Q,j})] \end{aligned}$$

is a sum of differences like that in (A), so

$$\psi(a_{I-P,i}, a_{J-Q,i}) \leqq \psi(a_{I-P,j}, a_{J-Q,j}). \tag{3}$$

On writing it out, this is the same as

$$\varphi(a_{N,i}) + \varphi(a_{N,j}) \leqq \varphi(b_{N,i}) + \varphi(b_{N,j}), \tag{4}$$

so

$$\sum_{r=1}^{m} \varphi(a_{N,r}) \leqq \sum_{r=1}^{m} \varphi(b_{N,r}). \tag{5}$$

If (A^*) holds, inequality (3) and hence (5) will be strict unless $P \cup Q \neq \emptyset$ or $a_{k,i} = a_{k,j}$ for all $k \in (I - P) \cup (J - Q)$.

There are $\mathbf{b}(1), \ldots, \mathbf{b}(q)$ such that $\mathbf{b}(1) = \mathbf{a}_N, \mathbf{b}(q) = (\mathbf{a}_I^*, \mathbf{a}_J')$ and for each $1 \leqq k \leqq n - 1$ there are i and j such that $\mathbf{b}(k+1) = S_{i,j}\mathbf{b}(k)$. Hence $\sum_{j=1}^m \varphi(b(1)_j) \leqq \ldots \leqq \sum_{j=1}^m \varphi(b(q)_j)$, which proves (1).

In (ii) it is clear that (b) $\Rightarrow$ (c) $\Rightarrow$ (a) always. We begin by assuming (A^*) holds and show that (a) $\Rightarrow$ (b). Suppose (b) does not hold. Then an examination of cases shows there are $1 \leqq i < j \leqq m$ such that for P and Q as above we have $P \cup Q \neq \emptyset$, and there is a $k \in (I - P) \cup (J - Q)$ such that $a_{k,i} \neq a_{k,j}$. Hence letting $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$ we have $\sum_{r=1}^m \varphi(a_{N,r}) < \sum_1^m \varphi(b_{N,r}) \leqq \sum_1^m \varphi(b_{I,r}^*, b_{J,r}') = \sum_1^m \varphi(a_{I,r}^*, a_{J,r}')$, since $\mathbf{b}_k^* = \mathbf{a}_k^*$, $k = 1, \ldots, n$. Conversely if (a) $\Rightarrow$ (b), then the arguments used in proving necessity of (A) for (1) show that (A^*) holds.

We turn now to the proof of (iii). Since $\varphi(\mathbf{a}_I^*, \mathbf{a}_J') \sim \varphi(\mathbf{a}_I', \mathbf{a}_J^*)$, it suffices to prove (2) assuming φ is increasing in the I-variables and decreasing in the J-variables. In this case let $\mathbf{b}_N = S_{i,j}\mathbf{a}_N$. Then

$$\varphi(b_{N,j}) \leqq \varphi(a_{N,i}), \ \varphi(a_{N,j}) \leqq \varphi(b_{N,i}). \tag{6}$$

We call $\varphi(a_{N,i})$ and $\varphi(a_{N,j})$ the "old terms", and $\varphi(b_{N,i})$ and $\varphi(b_{N,j})$ the "new terms". These are the only terms where $\varphi(\mathbf{a}_N)$ and $\varphi(\mathbf{b}_N)$ differ.

Let $1 \leqq k \leqq m$, define sequences

$$\boldsymbol{\alpha} = (\varphi(a_N)_r^* : 1 \leqq r \leqq k), \quad \boldsymbol{\beta} = (\varphi(b_N)_r^* : 1 \leqq r \leqq k),$$

let $\sum \boldsymbol{\alpha} = \sum_{r=1}^k \varphi(a_N)_r^*$ and define $\sum \boldsymbol{\beta}$ similarly. We show that $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If exactly one of the old terms occurs in $\boldsymbol{\alpha}$, then (6) implies that the only new term in $\boldsymbol{\beta}$ is $\varphi(b_{N,i})$. For if $\varphi(b_{N,j})$ is in $\boldsymbol{\beta}$, then (6) implies that $\boldsymbol{\beta}$ contains both new terms, so there are $m - k$ terms of $\varphi(\mathbf{a}_N)$ which are $\leqq \varphi(b_{N,j})$, in which case (6) implies that both old terms occur in $\boldsymbol{\alpha}$. Hence $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by replacing an old term by the larger term $\varphi(b_{N,i})$. Thus $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If both old terms occur in $\boldsymbol{\alpha}$, then (4) implies their sum is $\leqq$ the sum of the new terms, which is $\leqq$ the sum of $\varphi(b_{N,i})$ and any term $\geqq \varphi(b_{N,j})$, in case $\varphi(b_{N,j})$ is not in $\boldsymbol{\beta}$. Hence $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$.

If none of the old terms occur in $\boldsymbol{\alpha}$, then either $\boldsymbol{\alpha} = \boldsymbol{\beta}$, or $\boldsymbol{\beta}$ is obtained from $\boldsymbol{\alpha}$ by replacing one term of $\boldsymbol{\alpha}$ by the larger term $\varphi(b_{N,i})$. Thus $\sum \boldsymbol{\alpha} \leqq \sum \boldsymbol{\beta}$. The proof of (iii) is finished as in (i). This completes the proof of the theorem.

When φ is a function of two variables, conditions (A) and (A^*) simplify, and the arguments proving (3.1) have a symmetry which shows how small the sums can get.

(3.2) COROLLARY. *Let* $\varphi : T_1 \times T_2 \to G$.

(i) *The inequality*

$$\sum_{j=1}^m \varphi(a_j^*, b_j') \leqq \sum_{j=1}^m \varphi(a_j, b_j) \leqq \sum_{j=1}^m \varphi(a_j^*, b_j^*) \tag{1}$$

holds for all $\mathbf{a} \in (T_1)^m$ *and* $\mathbf{b} \in (T_2)^m$ *if and only if* $\Delta_{c,d}\varphi(y) = \varphi(d, y) - \varphi(c, y)$ *is increasing in* $y \in T_2$ *whenever* $d > c$, $d, c \in T_1$.

(ii) $\Delta_{c,d}\varphi$ *is strictly increasing whenever* $d > c$ *if and only if the following are equivalent:* (a) *Equality occurs in* (1) *on the left* (*right*)*;* (b) $\mathbf{a}$ *and* $\mathbf{b}$ *are oppositely* (*similarly*) *ordered;* (c) $\varphi(\mathbf{a}^*, \mathbf{b}') \sim \varphi(\mathbf{a}, \mathbf{b})$ $(\varphi(\mathbf{a}^*, \mathbf{b}^*) \sim \varphi(\mathbf{a}, \mathbf{b}))$.

(iii) *If the range of* φ *is totally ordered, and in addition to* (i), φ *is increasing* (*or decreasing*) *in both variables, then* $\varphi(\mathbf{a}^*, \mathbf{b}') \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}^*)$.

(3.3) *Remarks.* (i) In (3.2.i) above, replacing φ by $-\varphi$ gives the condition when the inequalities (1) reverse. The corresponding condition in (iii) is that φ be increasing in one variable and decreasing in the other, in which case, $\varphi(\mathbf{a}^*, \mathbf{b}^*) \ll \varphi(\mathbf{a}, \mathbf{b}) \ll \varphi(\mathbf{a}^*, \mathbf{b}')$.

(ii) The inequalities in (3.1), (3.2) and (3.3.i) may be written equivalently by interchanging primes and asterisks, since, for example, $\varphi(\mathbf{a}^*, \mathbf{b}') \sim \varphi(\mathbf{a}', \mathbf{b}^*)$.

4. Examples for the discrete case. In this section we illustrate the previous theorems for particular choices of φ. In all cases, $G = \mathbf{R}$.

(4.1) $T_1 = T_2 = \mathbf{R}$ and $\varphi(x, y) = x + y : \mathbf{a}^* + \mathbf{b}' \prec \mathbf{a} + \mathbf{b} \prec \mathbf{a}^* + \mathbf{b}^*$.

(4.2) $T_1 = T_2 = \mathbf{R}$ and $\varphi(x, y) = x - y : \mathbf{a}^* - \mathbf{b}^* \prec \mathbf{a} - \mathbf{b} \prec \mathbf{a}^* - \mathbf{b}'$.

(4.3) $\varphi(x, y) = xy$: For $T_1 = T_2 = \mathbf{R}$

we obtain (1.1) with the indicated condition for equality.

For $T_1 = T_2 = [0, \infty[$ or $T_1 = T_2 =]-\infty, 0]$ we obtain $\mathbf{a}^*\mathbf{b}' \ll \mathbf{ab} \ll \mathbf{a}^*\mathbf{b}^*$ whenever

$$\mathbf{a}, \mathbf{b} \in [0, \infty[^m \quad \text{or} \quad \mathbf{a}, \mathbf{b} \in]-\infty, 0]^m.$$

When $T_k = [0, \infty[$ $(k = 1, \ldots, n)$, $I = \{1, \ldots, n\}$ and $J = \emptyset$ then $\varphi(u_1, \ldots, u_n) = u_1 \ldots u_n$ satisfies (A) and we obtain a companion to (1.4), also proved by Ruderman:

$$\sum_{j=1}^{m} \prod_{i=1}^{n} a_{i,j} \leqq \sum_{j=1}^{m} \prod_{i=1}^{n} a_{i,j}^*.$$

If all $a_{i,j} > 0$, then the inequality is strict unless all of the sequences $\mathbf{a}_k = (a_{k,1}, \ldots, a_{k,m})$ are similarly ordered.

(4.4) $\varphi(x, y) = \log(1 + xy)$

with $T_1 \times T_2 \subset \{(x, y) : xy > -1\}$ gives:

$$\prod_{i=1}^{m} (1 + a_i^* b_i') \leqq \prod_{i=1}^{m} (1 + a_i b_i) \leqq \prod_{i=1}^{m} (1 + a_i^* b_i^*)$$

whenever $a_i^* b_i' > -1$ for $i = 1$ and $i = m$. The inequality is strict except as indicated in (3.2.ii). The choice $T_1 = T_2 = [0, \infty[$ or $]-\infty, 0]$ gives:

$$\log(1 + \mathbf{a}^*\mathbf{b}') \ll \log(1 + \mathbf{ab}) \ll \log(1 + \mathbf{a}^*\mathbf{b}^*)$$

whenever $\mathbf{a}, \mathbf{b} \in [0, \infty[^m$ or $]-\infty, 0]^m$.

(4.5) $\varphi(x, y) = -\log(x+y)$, $T_1 \times T_2 \subset \{(x, y) : x + y > 0\}$:
$$-\log(\mathbf{a}^* + \mathbf{b}') \ll -\log(\mathbf{a} + \mathbf{b}) \ll -\log(\mathbf{a}^* + \mathbf{b}^*)$$

whenever $a_m^* + b_m^* > 0$, and in particular we get an inequality of Minc [**10**]:

$$\prod_{i=1}^{m} (a_i^* + b_i^*) \leqq \prod_{i=1}^{m} (a_i + b_i) \leqq \prod_{i=1}^{m} (a_i^* + b_i').$$

The inequality is strict except as indicated by (3.2.i). The example $\mathbf{a} = (6, 5, 2, 1)$ $\mathbf{b} = (-3, -4, -2, 1)$ shows this inequality may fail under the condition $a_i + b_i \geqq 0$ for all i (but it will hold for vectors of length $\leqq 3$). This inequality is also easily seen to hold for all $a_i, b_i \geqq 0$.

Analogously, $\varphi(u_1, \ldots, u_n) = -\log(u_1 + \ldots + u_n)$ with

$$T_1 \times \ldots \times T_n \subset \{(u_1, \ldots, u_n) : u_1 + \ldots + u_n > 0\}$$

gives Ruderman's Inequality (1.2) whenever $\sum_{k=1}^{n} a_{k,m}^* > 0$. The inequality is strict unless all the $\mathbf{a}_k$ are similarly ordered.

(4.6) Suppose φ satisfies the hypotheses of (3.1.iii) and H is increasing and convex on an interval containing the range of φ. Then $\varphi_1 = H \circ \varphi$ satisfies condition (A). In this way [**11**, p. 165, Theorem 2] and (3.1.i) may be used to prove (3.1.iii). If in addition, φ satisfies (A^*) and H is strictly convex, then φ_1 satisfies (A^*). The proof follows easily from [**11**, p. 164, the third inequality from the bottom].

(4.7) Two theorems of D. London [**6**] may be obtained using (3.2) and (4.6). Replace a_i by $1/a_i$, so that his results are stated without quotients. His conditions on F in both theorems are the same as saying that F is convex and increasing on $[0, \infty[$. Hence let $H = F$, let $\varphi(x, y) = \log(1 + xy)$ for Theorem 1, and let $\varphi(x, y) = xy$ for Theorem 2. If F is strictly convex, we obtain his conditions for equality.

(4.8) Ruderman [**12**] has observed that (1.2) generalizes the inequality between the arithmetic and geometric means. Using (3.1) we may obtain the following inequality for certain quasi-arithmetic symmetric means. Let U be an open interval of $\mathbf{R}$, let $f, g : U \to \mathbf{R}$ be strictly monotone and let $f \circ g^{-1}$ be convex on $g[U]$. If f is increasing then

$$g^{-1}([g(r_1) + \ldots + g(r_n)]/n) \leqq f^{-1}([f(r_1) + \ldots + f(r_n)]/n)$$

for all $r_1, \ldots, r_n \in U$, while if f is decreasing, the inequality reverses. If $f \circ g^{-1}$ is strictly convex, the inequality is strict unless $r_1 = \ldots = r_n$. To prove this, in (3.1.i.1) let

$$\mathbf{a}_1 = (r_1, r_2, \ldots, r_{n-1}, r_n),$$
$$\mathbf{a}_2 = (r_2, r_3, \ldots, r_n, r_1), \ldots, \mathbf{a}_n = (r_n, r_1, \ldots, r_{n-2}, r_{n-1})$$

and note that

$$\varphi(u_1, \ldots, u_n) = f \circ g^{-1}([g(u_1) + \ldots + g(u_n)]/n)$$

satisfies (A) with $I = \{1, \dots, n\}$. If $f \circ g^{-1}$ is strictly convex, then φ satisfies (A^*), and the inequality is strict unless all the $\mathbf{a}_k$ are similarly ordered, in which case $r_1 = \dots = r_n$.

(4.9) For $\varphi(x, y) = (x + y)^p$ with real $p > 0$ we have:

(i) $(\mathbf{a}^* + \mathbf{b}')^p \ll (\mathbf{a} + \mathbf{b})^p \ll (\mathbf{a}^* + \mathbf{b}^*)^p$ if $p > 1$,

(ii) $\sum_{j=1}^{m} (a_j^* + b_j^*)^p \leqq \sum_{j=1}^{m} (a_j + b_j)^p \leqq \sum_{j=1}^{m} (a_j^* + b_j')^p$ if $p < 1$,

whenever $a_m^* + b_m^* \geqq 0$. The inequalities are strict except as indicated in (3.2) and (3.3). If p is an integer, then (i) holds for all $\mathbf{a}, \mathbf{b} \in \mathbf{R}^m$. The example $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (3, 1, 2)$ shows that relation $\ll$ cannot be used in (ii).

5. The continuous case. In this section we show how to generalize Theorems (3.1) and (3.2) for L^∞ functions on a finite measure space (X, Λ, μ) when φ is jointly continuous. If $f_1, \dots, f_n \in L^\infty$ and $\varphi : I_{f_1} \times \dots \times I_{f_n} \to \mathbf{R}$ is bounded, then the function $\varphi(f_1, \dots, f_n)$ defined by $x \mapsto \varphi(f_1(x), \dots, f_n(x))$ is in L^∞. If $\{I, J\}$ is a partition of $\{1, \dots, n\}$ then $(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J})$ denotes $(g_1, \dots, g_n)$ where $g_i = \delta_{f_i}$ for $i \in I$ and $g_i = \iota_{f_i}$ for $i \in J$.

(5.1) Theorem. *Let* $\varphi : T_1 \times \dots \times T_n \to \mathbf{R}$ *be continuous, where* $T_1, \dots, T_n$ *are intervals of* $\mathbf{R}$, *and let* $\{I, J\}$ *be a partition of* $\{1, \dots, n\}$.

(i) *If* φ *satisfies condition* (A) *then*

$$\int \varphi(f_1, \dots, f_n) d\mu \leqq \int_0^\alpha \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J}) \tag{1}$$

for all $f_i \in L^\infty$ *such that* $I_{f_i} \subset T_i$, $i = 1, \dots, n$. *If* (X, Λ, μ) *is non-atomic, then* (A) *is necessary for* (1).

(ii) *If* φ *satisfies* (A^*) *then the following are equivalent:*

(a) *Equality holds in* (1).

(b) f_i *and* f_j *are similarly ordered whenever* i *and* j *are in the same set* I *or* J, *and oppositely ordered whenever* i *and* j *are in different sets* I *and* J *for all* $1 \leqq i, j \leqq n$.

(c) $\varphi(f_1, \dots, f_n) \sim \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J})$.

(iii) *If* φ *satisfies* (A) *and is increasing* (*respectively decreasing*) *in* u_i *for* $i \in I$ *and decreasing* (*respectively increasing*) *for* $i \in J$, *then for all* f_i *as in* (i) *we have*

$$\varphi(f_1, \dots, f_n) \ll \varphi(\delta_{\mathbf{f}_I}, \iota_{\mathbf{f}_J}).$$

(5.2) Corollary. *Let* $\varphi : T_1 \times T_2 \to \mathbf{R}$ *be continuous, where* T_1 *and* T_2 *are intervals of* $\mathbf{R}$, *and let* $f, g \in L^\infty$ *with* $I_f \subset T_1$ *and* $I_g \subset T_2$.

(i) *If* $\Delta_{c,d}\varphi(y)$ *is increasing in* $y \in T_2$ *whenever* $d > c$ *and* $d, c \in T_1$, *then*

$$\int_0^\alpha \varphi(\delta_f, \iota_g) \leqq \int \varphi(f, g) d\mu \leqq \int_0^\alpha \varphi(\delta_f, \delta_g). \tag{1}$$

(ii) *If* $\Delta_{c,d}\varphi$ *is strictly increasing, then the following are equivalent:* (a) *Equality occurs in* (1) *on the left* (*right*); (b) *f and g are oppositely* (*similarly*) *ordered;* (c) $\varphi(\delta_f, \iota_g) \sim \varphi(f, g)$ $(\varphi(\delta_f, \delta_g) \sim \varphi(f, g))$.

(iii) *If in addition to* (i) φ *is increasing in both variables or decreasing in both variables, then*

$$\varphi(\delta_f, \iota_g) \ll \varphi(f, g) \ll \varphi(\delta_f, \delta_g). \tag{2}$$

(5.3) *Remark.* The conditions that the inequalities in (5.2) reverse are the same as in (3.3). If φ satisfies these conditions, then (5.2) may be applied to $\varphi_1(x, y) = \varphi(x, r + s - y)$, f, and $g_1 = r + s - g$, where $I_g = [r, s]$.

We begin by showing that it suffices to prove (5.1) and (5.2) for non-atomic measure spaces by embedding (X, Λ, μ) in a non-atomic m.s. $(X^\#, \Lambda^\#, \mu^\#)$. (See [**9**] or [**2**] for details of this method.) If $f \in M(X, \mu)$, then the corresponding member of $M(X^\#, \mu^\#)$ is denoted by $f^\#$. Then $\varphi(f_1^\#, \ldots, f_n^\#) = \varphi(f_1, \ldots, f_n)^\# \sim \varphi(f_1, \ldots, f_n)$. In addition it is not hard to see that f and g are similarly (oppositely) ordered if and only if $f^\#$ and $g^\#$ are similarly (oppositely) ordered. Thus if (5.1) and (5.2) are true when (X, Λ, μ) is non-atomic, then they are true for any finite m.s.

Before proceeding with the proof when (X, Λ, μ) is non-atomic, we require some lemmas. The first two are needed when the measure space is not separable, for otherwise it is measure theoretically $[0, \alpha]$.

(5.4) LEMMA. *Let* (X, Λ, μ) *be non-atomic. Suppose* $\{D_k\}_{k=1}^N$ *is a partition of X by measurable sets. If* $\epsilon > 0$, *then there is a partition* $\{E_i\}_{i=1}^n$ *of X by measurable sets such that* $\mu(E_i) = \mu(X)/n$ $(i = 1, \ldots, n)$ *and* $\mu(\bigcup\{E_i : E_i$ *intersects more than one* $D_k\}) < \epsilon$.

Proof. Let $\alpha = \mu(X)$. If $\alpha = 0$, the lemma is trivially true. Otherwise, rename the sets D_k so that $\mu(D_k) = 0$ for $1 \leqq k < p$ and $\mu(D_k) > 0$ for $p \leqq k \leqq N$. There is a $\phi : [0, \alpha] \to \Lambda$ such that $\mu(\phi(t)) = t$, $t \leqq u$ implies $\phi(t) \subset \phi(u)$, $\phi(0) = \bigcup_{1 \leqq k < p} D_k$, and $\phi(\sum_{1 \leqq k \leqq q} \mu(D_k)) = \bigcup_{1 \leqq k \leqq q} D_k$ for $q = p, \ldots, N$ (use [**2**, (5.6)]). For any n such that $\alpha/n \leqq \min\{\mu(D_k) : p \leqq k \leqq N\}$ and for $E_i = \phi(\alpha i/n) - \phi(\alpha(i-1)/n)$ $(i = 1, \ldots, n)$ we have that each E_i intersects at most two sets D_k of positive measure, and at most N $- 1$ of these E_i intersect more than one D_k. To finish the proof, choose n so that also $\alpha(N-1)/n < \epsilon$.

(5.5) LEMMA. *Suppose* (X, Λ, μ) *is non-atomic. Let* $\{s(k)_i\}_{i=1}^\infty$ $(k = 1, \ldots, n)$ *be n sequences of simple functions. Then there are n sequences* $\{t(k)_i\}_{i=1}^\infty$, $(k = 1, \ldots, n)$ *of simple functions such that*

(i) *For each* i, $t(1)_i, \ldots, t(n)_i$ *have the same sets of constancy, and these sets have equal measure;*

(ii) *For each* $k = 1, \ldots, n$, $s(k)_i - t(k)_i \to 0$ μ*-almost everywhere as* $i \to \infty$;

(iii) *For each* $k = 1, \ldots, n$ *and* $i \geqq 1$, $|t(k)_i| \leqq |s(k)_i|$.

Proof. For clarity of exposition, we prove the lemma in the case $n = 2$. The proof for larger n will be readily apparent. Before considering sequences, let $s(1) = \sum_{i=1}^{n} a_i 1_{A_i}$ and $s(2) = \sum_{j=1}^{p} b_j 1_{B_j}$ where $\{A_i\}$ and $\{B_j\}$ partition X, and let $\{D_k\}_{k=1}^{N} = \{A_i \cap B_j : 1 \leqq i \leqq n,\ 1 \leqq j \leqq p\}$. Let $\epsilon > 0$. Then there is a measurable partition $\{E_q\}_{q=1}^{r}$ as in Lemma (5.4). For each $q = 1, \ldots, r$, if E_q intersects only $A_i \cap B_j$ then $E_q \subset A_i \cap B_j$, and for $k = 1, 2$ we define $t(k)|E_q = s(k)|(A_i \cap B_j)$; we define $t(k) = 0$ elsewhere. Then $|t(k)| \leqq |s(k)|$ and $\mu(\{s(k) \neq t(k)\}) < \epsilon$. Hence given $\{s(k)_i\}_{i=1}^{\infty}$ there are sequences $\{t(k)_i\}_{i=1}^{\infty}$ satisfying (i) and (iii) such that $\mu(\{s(k)_i \neq t(k)_i\}) < 2^{-i}$. Then

$$\mu(\{s(k)_i - t(k)_i \nrightarrow 0\}) = \\ \mu\left(\bigcup_{q=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{i=N}^{\infty} \{|s(k)_i - t(k)_i| > 1/q\}\right) \leqq \lim_{q\to\infty} \lim_{N\to\infty} \sum_{i=N}^{\infty} 2^{-i} = 0,$$

and the proof is finished.

(5.6) Proposition. *Suppose* (X, Λ, μ) *is non-atomic, let* $f_1, \ldots, f_n \in M(X, \mu)$, *let* $\{I, J\}$ *be a partition of* $\{1, \ldots, n\}$, *and let* $F_1, \ldots, F_n \in [0, \alpha]$ *with* F_i *right continuous and decreasing (increasing) when* $i \in I$ $(i \in J)$. *Then the following three conditions are equivalent.*

(i) $(f_1, \ldots, f_n) \sim (F_1, \ldots, F_n)$.

(ii) *There is a measure preserving map* $\sigma : X \to [0, \alpha]$ *such that* $F_i \circ \sigma = f_i$ μ*-almost everywhere,* $1 \leqq i \leqq n$.

(iii) f_i *and* f_j *are similarly ordered when* i *and* j *are in the same set* I *or* J, *oppositely ordered when* i *and* j *are in different sets* I *and* J, *and* $F_i = \delta_{f_i}$ *for* $i \in I$, $F_j = \iota_{f_j}$ *for* $j \in J$.

Proof. Let $A \subseteq B[\mu]$ mean $\mu(A \backslash B) = 0$, i.e., $1_A \leqq 1_B$ μ-almost everywhere. Writing $\mathbf{f} = (f_1, \ldots, f_n)$, $\mathbf{F} = (F_1, \ldots, F_n)$, and $\mathbf{t} = (t_1, \ldots, t_n)$, the proof given in [**2**, Theorem (6.2)] shows (i) $\Rightarrow$ (ii). Also, (ii) $\Rightarrow$ (iii) is straightforward.

We prove (iii) $\Rightarrow$ (i) first in the case $J = \emptyset$.

I. If f and g are similarly ordered, then for all $t \in \mathbf{R}$, $\operatorname{ess.sup} g|\{f \leqq t\} \leqq \operatorname{ess.inf} g|\{f > t\}$. This follows from $\operatorname{ess.sup} g|\{f > t + 1/n\} \to \operatorname{ess.sup} g|\{f > t\}$ as $n \to \infty$.

II. If f and g are similarly ordered, and $t, u \in \mathbf{R}$, then $\{f > t\} \cap \{g > u\} = \{f > t\}$ or $\{g > u\}$ $[\mu]$. Indeed, let

$$A = \{f \leqq t\} \cap \{g > u\}, \qquad B = \{f > t\} \cap \{g \leqq u\},$$

and suppose both $\mu(A), \mu(B) > 0$. Then $\operatorname{ess.inf} g|B \leqq u < \operatorname{ess.sup} g|A$, while (I) implies $\operatorname{ess.sup} g|A \leqq \operatorname{ess.inf} g|B$, a contradiction. Hence $\mu(A) = 0$ or $\mu(B) = 0$.

III. If $\{f > t\} \subset \{g > u\}$ $[\mu]$ then $\{\delta_f > t\} \subset \{\delta_g > u\}$. Indeed, $\{\delta_f > t\} = [0, \mu\{f > t\}[\subset [0, \mu\{g > u\}[= \{\delta_g > u\}$.

IV. It follows from (II) and (III) that for all $\mathbf{t} \in \mathbf{R}^n$, $\mu\{\mathbf{f} > \mathbf{t}\} = \mu(\bigcap \{f_i > t_i\}) = m(\bigcap \{\delta_{f_i} > t_i\}) = m\{\mathbf{F} > \mathbf{t}\}$, so $\mathbf{f} \sim \mathbf{F}$.

To deduce the general case from this one, let $\varphi(t_1, \ldots, t_n) = (u_1, \ldots, u_n)$, where $u_i = t_i$ if $i \in I$, $= -t_i$ if $i \in J$, let $(f_1', \ldots, f_n') = \varphi(f_1, \ldots, f_n)$, and let $F_i' = \delta_{f_i'}$. By the $J = \emptyset$ case, $\mathbf{F}' \sim \mathbf{f}'$, so $\mathbf{F} = \varphi(\mathbf{F}') \sim \varphi(\mathbf{f}') = \mathbf{f}$ (because $\delta_{-f} = -\iota_f$).

We can now prove (5.1) and (5.2). For clarity of exposition we will only present a proof of (5.2). The proof of (5.1) will then be clear. With regard to (5.1.ii) we remark that (5.6) shows that (b) $\Rightarrow$ (c) $\Rightarrow$ (a) always. The proof of (5.2) will illustrate the proof of (a) $\Rightarrow$ (b) when $n = 2$.

Proof of (5.2). Let $v = \sum_{j=1}^{m} a_j 1_{E_j}$ and $w = \sum_{j=1}^{m} b_j 1_{E_j}$, where $a_j \in T_1$, $b_j \in T_2$ $(1 \leqq j \leqq m)$ and $\mu(E_j) = \alpha/m$. In case (i), (3.2.i) gives

$$(*) \qquad \int_0^\alpha \varphi(\delta_v, \iota_w) \leqq \int \varphi(v, w)\,d\mu \leqq \int_0^\alpha \varphi(\delta_v, \delta_w)$$

while in case (ii), (3.2.iii) gives for $t = k\alpha/p$ $(1 \leqq k \leqq m)$

$$(**) \qquad \int_0^t \delta_{\varphi(\delta_v, \iota_w)} \leqq \int_0^t \delta_{\varphi(v,w)} \leqq \int_0^t \delta_{\varphi(\delta_v, \delta_w)}.$$

Now in (**) each of the integrands is constant on each of the intervals $[(j-1)\alpha/n, j\alpha/n[$, so the integrals are linear functions of t on these intervals, and hence (**) holds for all $0 \leqq t \leqq \alpha$. Using now (5.5) there are sequences v_i and w_i of simple functions like v and w above such that $v_i \to f$, $w_i \to g$, $|v_i| \leqq |f|$ and $|w_i| \leqq |g|$, so $\delta_{v_i} \to \delta_f$ and $\delta_{w_i} \to \delta_g$ almost everywhere. Since φ is bounded on $I_f \times I_g$, each integrand in (*) or (**) is bounded by a constant depending only on f and g. Taking limits and using the dominated convergence theorem, we have that (*) or (**) holds with v and w replaced by f and g respectively.

We now show the condition for equality on the right in (3.2.i.1). Assume φ satisfies (A^*), suppose f and g are not similarly ordered, and we will show that the inequality on the right is strict. There are disjoint sets A and B of positive measure such that

$$\text{ess.sup}\, f\,|A < \text{ess.inf}\, f\,|B \qquad \text{and} \qquad t = \text{ess.sup}\, g|A > \text{ess.inf}\, g|B = r.$$

Let $r < s_1 < s_2 < t$ and let

$$D \subset \{x \in A : g(x) \geqq s_2\} \qquad \text{and} \qquad E \subset \{x \in B : g(x) \leqq s_1\}$$

with $0 < \mu(D) = \mu(E) = \beta$. Then let $\sigma_D : D \to [0, \beta[$ and $\sigma_E : E \to [0, \beta[$ be measure preserving and define

$$f' = \delta_{f|D} \circ \sigma_D \text{ on } D, = \delta_{f|E} \circ \sigma_E \text{ on } E, \text{ and } = f \text{ elsewhere;}$$
$$g' = \delta_{g|E} \circ \sigma_D \text{ on } D, = \delta_{g|D} \circ \sigma_E \text{ on } E, \text{ and } = g \text{ elsewhere.}$$

Then $f' \sim f$, $g' \sim g$, $\delta_{f|D} < \delta_{f|E}$, and $\delta_{g|E} < \delta_{g|D}$. Hence

$$\begin{aligned}\int_D \varphi(f, g)d\mu + \int_E \varphi(f, g)d\mu &\leqq \int_0^\beta [\varphi(\delta_{f|D}, \delta_{g|D}) + \varphi(\delta_{f|E}, \delta_{g|E})] \\ &< \int_0^\beta [\varphi(\delta_{f|D}, \delta_{g|E}) + \varphi(\delta_{f|E}, \delta_{g|D})] \\ &= \int_D \varphi(f', g')d\mu + \int_E \varphi(f\ , g')d\mu.\end{aligned}$$

Adding

$$\int_{X-(D \cup E)} \varphi(f, g)d\mu = \int_{X-(D \cup E)} \varphi(f', g')d\mu$$

we obtain

$$\int \varphi(f, g)d\mu < \int \varphi(f', g')d\mu \leqq \int_0^\alpha \varphi(\delta_{f'}, \delta_{g'}) = \int_0^\alpha \varphi(\delta_f, \delta_g),$$

and the proof is finished.

(5.7) *Remark.* Depending on the choice of φ and the intervals T_i, Theorems (5.1) and (5.2) may hold for a larger set of functions than L^∞. Indeed, the proof shows that in (5.2) inequalities (1) or (2) will hold whenever limit and integral can be interchanged in (*) or (**). The condition for equality holds if (5.2.1) holds for $f|A$ and $g|A$ for all $A \in \Lambda$ whenever it holds for f and g.

For example, suppose $f_1, \ldots, f_m \in L^p$ implies $\varphi(f_1, \ldots, f_m) \in L^1$. Now it follows from [**9**, p. 93] that $|v| \leqq |f|$ implies $|\delta_v| \leqq |\delta_f|$ and $|\iota_v| \leqq |\iota_f|$, so we may use [**3**] and the dominated convergence theorem to conclude that (5.1.1) and (5.2.1) hold for all L^p functions. Finally, since $f_1, \ldots, f_m \in L^p$ implies $f_1|A, \ldots, f_m|A \in L^p$, the condition for equality also holds for all L^p functions. Other illustrations appear in the following examples.

6. Examples for the continuous case.

(6.1) (i) $\delta_f + \iota_g \prec f + g \prec \delta_f + \delta_g$ for all $f, g \in L^1$.

(ii) $\delta_f - \delta_g \prec f - g \prec \delta_f - \iota_g$ for all $f, g \in L^1$.

The (i) and (ii) are easily seen to be equivalent using [**9**, p. 93]. While $\delta_{f+g} \prec \delta_f + \delta_g$ is well-known (see [**9**, p. 108]), the fact that $\delta_f - \delta_g \prec f - g$ is new. Then a theorem of Luxemburg [**9**, p. 107] implies $|\delta_f - \delta_g| \ll |f - g|$, generalizing [**8**, Proposition 1, p. 34]. It then follows that $\|f_\beta - f\|_1 \to 0$ implies $\|\delta_{f_\beta} - \delta_f\|_1 \to 0$, where $\{f_\beta\}$ is a net. Using [**9**, (9.1)], the inequality $\delta_f - \delta_g \prec f - g$ can be written equivalently:

$$\int_E \delta_f + \int_E \delta_g(\alpha - t)dt \leqq \int_0^{m(E)} \delta_{f+g}$$

for all Lebesgue measurable $E \subset [0, \alpha]$, where m denotes Lebesgue measure. This is an interesting generalization of [**9**, (10.1)].

(6.2) An inequality of Hardy-Littlewood-Polya-Luxemburg:

$$\int_0^\alpha \delta_f \iota_g \leqq \int fg \, d\mu \leqq \int_0^\alpha \delta_f \delta_g$$

holds for all $f, g \in L^\infty$, and, using monotone convergence, it is easily seen to hold for all $0 \leqq f, g \in M$. Then as in [**9**, p. 102], it may be shown to hold whenever $\delta_{|f|}\delta_{|g|} \in L^1[0, \alpha]$. The inequalities are strict except as indicated in (5.2). Similarly, $\delta_f \iota_g \ll fg \ll \delta_f \delta_g$ for all $0 \leqq f, g \in M$ such that $\delta_f \delta_g \in L^1[0, \alpha]$.

$$\text{(6.3) (i)} \quad \int_0^\alpha \log(1 + \delta_f \iota_g) \leqq \int \log(1 + fg) d\mu \leqq \int_0^\alpha \log(1 + \delta_f \delta_g)$$

holds for all $f, g \in L^\infty$ satisfying both

$$\text{(ii)} \qquad \delta_f(0)\iota_g(0) > -1 \text{ and } \delta_f(\alpha-)\iota_g(\alpha-) > -1,$$

because (ii) is equivalent to: $I_f \times I_g \subset \{(x, y) : xy > -1\}$. In addition, using monotone convergence, (i) can be shown to hold if $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$. Then (i) can be shown to hold for all $f, g \in M$ satisfying (ii) using the following observations. First, $\log(1 + fg) = \log(1 + f^+g^+) + \log(1 - f^+g^-) + \log(1 - f^-g^+) + \log(1 + f^-g^-)$. Next, when (ii) holds for the pair f, g it also holds for each of the pairs: f^+, g^+; $f^+, -g^-$; $-f^-, g^+$; $-f^-, -g^-$. Finally, when (ii) holds, then: f unbounded above implies $g \geqq 0$; f unbounded below implies $g \leqq 0$; and the same is true when f and g are interchanged. Clearly if $f, g \in M$ satisfy (ii) so do $f|A$ and $g|A$ for any $A \in \Lambda$. Hence the inequalities are strict as indicated in (5.2).

Similarly, $\log(1 + \delta_g \iota_g) \ll \log(1 + fg) \ll \log(1 + \delta_f \delta_g)$ for all $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$ such that $\log(1 + \delta_f \delta_g) \in L^1[0, \alpha]$.

$$\text{(6.4) (i)} \quad \int_0^\alpha \log(\delta_f + \delta_g) \leqq \int \log(f + g) d\mu \leqq \int_0^\alpha \log(\delta_f + \iota_g)$$

for all $f, g \in L^\infty$ such that

$$\text{(ii)} \qquad \delta_f(\alpha-) + \delta_g(\alpha-) > 0,$$

since (ii) is equivalent to $I_f \times I_g \subset \{(x, y) : x + y > 0\}$. Actually, (i) holds for all $f, g \in M$ satisfying (ii) since f and g are then bounded below, so we may approximate them by increasing sequences of bounded functions satisfying (ii) and use the B. Levi monotone convergence theorem [**5**, p. 172]. The inequalities are strict except as indicated in (5.3). Similarly, if $f, g \in M$ satisfy (ii) and $\log(\delta_f + \iota_g) \in L^1[0, \alpha]$ then $-\log(\delta_f + \iota_g) \ll -\log(f + g) \ll -\log(\delta_f + \delta_g)$.

(6.5) We have the following continuous version of London's Theorems. Suppose $0 \leqq f, g \in M$ or $0 \geqq f, g \in M$.

(i) If H is convex, increasing and continuous on $[0, \infty[$, then

$$\int_0^\alpha H(\delta_f\iota_g) \leqq \int H(fg)\,d\mu \leqq \int_0^\alpha H(\delta_f\delta_g).$$

(ii) If $H(e^x)$ is convex, increasing and continuous on $[0, \infty[$, then

$$\int_0^\alpha H(1+\delta_f\iota_g) \leqq \int H(1+fg)\,d\mu \leqq \int_0^\alpha H(1+\delta_f\delta_g).$$

In either case, if H is strictly convex, then we have equality on the left (right) if and only if f and g are oppositely (similarly) ordered if and only if $\delta_f\iota_g \sim fg$ ($\delta_f\delta_g \sim fg$).

(6.6) For real $p > 0$ we have:

(i) $(\delta_f + \iota_g)^p \ll (f+g)^p \ll (\delta_f + \delta_g)^p$ if $p > 1$,

(ii) $$\int_0^\alpha (\delta_f + \delta_g)^p \leqq \int (f+g)^p\,d\mu \leqq \int_0^\alpha (\delta_f + \iota_g)^p \text{ if } p < 1,$$

whenever (a) $\delta_f(\alpha-) + \delta_h(\alpha-) \geqq 0$ and $f, g \in L^p$; or (b) $0 \leqq f, g \in M$; or (c) p is an integer and $f, g \in L^p$. The (i) gives a lower bound to an inequality of Chong and Rice [**2**, p. 88]. The inequalities are strict except as indicated in (5.2) and (5.3).

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AN EMBEDDING THEOREM FOR ORDERED GROUPS

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Definition. An *O^*-group* is a group wherein every partial order can be extended to some full order.

THEOREM. *Suppose the group G has a normal chain $G = G_1 \supseteq G_2 \supseteq \dots$ such that*

$$\bigcap_{i=1}^{\infty} G_i = E \text{ (identity subgroup)}$$

and each G/G_i is locally nilpotent and torsion-free. Then G can be embedded in the complete direct product G' of divisible O^*-groups.

Proof. Consider this map f of G into the complete direct product H of the G/G_i:

$$f(g) = (G_1g, G_2g, \dots) \text{ for every } g \text{ in } G.$$

Clearly f is a homomorphism and is one-to-one since $\bigcap_{i=1}^{\infty} G_i = E$; thus $G \cong f(G)$. Since each G/G_i is torsion-free and locally nilpotent, it can be embedded in a (unique minimal) divisible locally nilpotent torsion-free group D_i, where D_i consists of all roots of elements of G/G_i (cf. [**5**, p. 256]). Each D_i, being torsion-free and locally nilpotent, is O^* by [**6**, p. 174], and the theorem follows by taking G' as the complete direct product of the D_i. (The theorem is immediate if any $G_i = E$, since G itself is then a locally nilpotent torsion-free group.)

Definition. By a *central series* for a group G with identity subgroup E we mean a countably infinite normal chain (i.e., chain of subgroups normal in G):

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \dots$$

such that $\bigcap_{i=1}^{\infty} G_i = E$ and G_{i-1}/G_i is in the centre of G/G_i for every i.

Remark. Clearly the theorem holds for such G if also the factors G/G_i are torsion-free.

Now if F is any free group and $F = F_1 \supseteq F_2 \supseteq \dots$ is the lower central chain for F, then $\bigcap_{i=1}^{\infty} F_i = E$ by Magnus' theorem (cf. [**5**, p. 38]). Also, each F/F_i (the free nilpotent group of class i) is torsion-free; for if $a \notin F_i$ but $a^n \in F_i$ for nonzero n, then F_{s-1}/F_s is not torsion-free if $s \leqq i$ is chosen so that $a \in F_{s-1} - F_s$; however, Witt's theorem (cf. [**5**, p. 41]) states F_{s-1}/F_s

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is free abelian (and hence torsion-free; also, cf. [7]). Thus by the remark above, the theorem holds for F.

Necessary and sufficient conditions that a prescribed full order for a subgroup of a group extend to some full order for the group are unknown. However, if $G \subseteq G'$ are as in the theorem, and if no G_i is the identity E, then at least $2^{\aleph}$ full orders for G extend to full orders for G'. For each D_i $(i \geqq 2)$ admits at least two full orders, and the resulting uncountably many lexicographic full orders on G' are distinct on $H \subseteq G'$ since each D_i consists only of roots of elements of G/G_i. Hence, if no G_i is E, these orders are distinct on $G \cong f(G) \subseteq H$ by definition of f. (Of course, these remarks hold for free groups of rank $\geqq 2$, since no F_i is E.)

Definitions. A partial order P for a group G is called *isolated* if, for all x in G and all $n > 0$, $x^n \geqq e$ implies $x \geqq e$, where e is the group identity. A group in which every partial order extends to an isolated partial order will be called an *I^*-group*, following [**2**]. Likewise a group in which every right partial order extends to an isolated right partial order (obvious meaning) will be called RI^*, and one in which every right partial order extends to a right full order will be called RO^*.

In [**2**, p. 468] Hollister shows that a free group of rank $\geqq 2$ is not I^*. Also, an RI^*-group is I^* (for if a partial order P is contained in the isolated right partial order Q, then $\bigcap_{x \in G} x^{-1}Qx$ is an isolated partial order containing P), and so a nonabelian free group is not RI^*; cf. [**1**, pp. 69–70]. These results and the isomorphism f applied to F (notation as above) yield two corollaries.

COROLLARY 1. *A subdirect product of O^* (hence I^*) groups need not be I^* (hence not RI^* nor RO^*).*

Finally, any right partial order for F is a right partial order for $H \supseteq F$, where $H \equiv \prod F/F_i$ (complete direct product as in the proof of the theorem); thus H is not RI^* or RO^*. Also, in [**3**] Kargapolov showed that H is not O^*. Thus we have:

COROLLARY 2. *The complete direct product of O^*-groups need be neither RI^* (hence not RO^*) nor O^*.*

Note. It is unknown whether the classes I^*, RO^*, RI^* are closed under direct products. (In [**3**] and [**4**] the writers show that the restricted direct product of O^*-groups is O^*, whereas in [**8**] it is shown that the direct product of V^* (respectively V) groups is not V; a V^* (respectively V) group is one wherein every partial (respectively full) order on every subgroup extends to some full order for the group.)

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EXTENSIONS OF HOMOMORPHISMS OF PARABOLIC SUBGROUPS

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1. Introduction and definitions. If one wishes to construct a homomorphism of a group, an obvious approach is to begin with a homomorphism of a subgroup and attempt to extend it in some way. Any such extended homomorphism is determined by its action on elements of the group which, together with the subgroup, generate the whole group. The values assigned to such generators must then satisfy various functional equations. By a *governing list* for the extension problem, we shall mean a list of functional equations whose solutions describe all possible homomorphisms extending a given subgroup homomorphism.

Let G be a group and B a subgroup with presentation $(R; X)$, i.e., R is a set of generators for B and X is a set of relations on R which define B. Let S be a set of elements in G such that $R \cup S$ is a set of generators for G. A governing list for the pair (G, B) corresponds to a set of relations Y on $R \cup S$ such that $(R \cup S; X \cup Y)$ is a presentation of G. Since the approach here is directed towards extending a homomorphism of B to a homomorphism of G, we shall usually take $(R; X)$ to be the trivial group-table presentation of B.

Our primary interest is in obtaining governing lists convenient from the point of view of representation theory. To illustrate our criteria of desirability, let G, B and S be as above, and suppose that $f : B \rightarrow H$ is a homomorphism. Select a set of elements $H_S = \{h_s : s \in S\}$. Suppose there exists a homomorphism $\tilde{f} : G \rightarrow H$ with $\tilde{f}(b) = f(b)$ for all $b \in B$ and $\tilde{f}(s) = h_s$ for all $s \in S$. For each $s \in S$, define $N_s = B \cap s^{-1}Bs$. Then for $n \in N_s$, we have the functional relation

$$\tilde{f}(sn) = h_s f(n) = f(sns^{-1})h_s.$$

In other words, h_s intertwines the homomorphisms $n \rightarrow f(n)$ and $n \rightarrow f(sns^{-1})$ of N_s. Conditions of this form in a governing list will be denoted by (I) and will be referred to as *intertwining conditions*. If $s_i \in S$ and $b_i \in B$ $(i = 1, 2, 3)$ with $s_1b_1s_2 = b_2s_3b_3$, then we must also have $h_{s_1}f(b_1)h_{s_2} = f(b_2)h_{s_3}f(b_3)$. Conditions of this form involving three letter words in the generators of G will be referred to as triple conditions and will be denoted by (T). In representation theory, *triple conditions* are often much more difficult to analyze than intertwining conditions. An attempt is made here to exhibit governing lists involving a minimal number of triple conditions.

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A more obvious criterion of desirability is of course the size of the set S. If S is taken to be a set of (B, B) double coset representatives of G, it is straightforward to check that the extension process is governed by an intertwining condition for each $s \in S$ and triple conditions involving a set of representatives of $(N_s, N_{s'})$ double cosets of B for each pair $(s, s') \in S \times S$. In section 2 we investigate a case when S may be taken to be a singleton set. In sections 3 and 4, we investigate a case when the double coset space $W = B\backslash G/B$ has a group structure. In particular, this situation arises when G is a semi-simple Lie group and B a parabolic subgroup. W is commonly called the Weyl group of the pair (G, B). In addition to intertwining and triple conditions, the associated governing lists contain equations arising from a presentation of W. Such conditions will be denoted by (W).

2. Governing lists for subgroups with large double coset. Let G be a group and A a subset of G. Set $A^{-1} = \{g \in G : g^{-1} \in A\}$ and for $x \in G$, $Ax = \{g \in G : gx^{-1} \in A\}$. A is said to be *large* in G if $A^{-1} \cap Ax \cap Ay \cap Az \neq \emptyset$ for all x, y, z in G. If G is a locally compact group and dx is a (right or left) Haar measure on G, then a subset is large in G if its complement has zero measure.

Let A be large in G and let f be a mapping of A into a group H. Weil has shown [**4**] that f extends to a homomorphism of G into H if and only if $f(a) = f(a_1)f(a_2)$ whenever a, a_1, and a_2 are in A and $a = a_1a_2$.

Let G be a group, B a subgroup and suppose there exists $p \in G$ such that $A = BpB$ is large in G. Let $C = B \cap p^{-1}Bp$, $\tilde{C} = pCp^{-1}$, and $B = \bigcup_{j\in J} Cv_j\tilde{C}$ a union of distinct double cosets. Let $J' = \{j \in J : Cv_j\tilde{C} \subset A\}$. For each $j \in J'$ select elements v_j' and v_j'' in B such that $pv_jp = v_j'pv_j''$.

THEOREM 1. *With the notations above, let* $f : B \rightarrow H$ *be a homomorphism. Then* f *extends to a homomorphism* $\tilde{f} : G \rightarrow H$ *with* $\tilde{f}(p) = h$ *if and only if* h *satisfies the following list of conditions*:

(I) $hf(c) = f(\tilde{c})h$ *for* $c \in C$ *and* $\tilde{c} = pcp^{-1}$.

(T) $hf(v_j)h = f(v_j')hf(v_j'')$ *for* $j \in J'$.

When h *satisfies* (I) *and* (T), $\tilde{f}$ *is uniquely determined by* $\tilde{f}(p) = h$.

Proof. Necessity of the conditions (I) and (T) is obvious. It is immediate from the definition of a large subset that $A^2 = G$. Hence G is generated by B and p and the uniqueness statement is trivial.

Suppose h satisfies (I) and (T). Check that $b_1pb_2 = b_1'pb_2'$ implies that $b_2(b_2')^{-1} \in C$. From (I), it follows that for $a = b_1pb_2 \in A$, $a \rightarrow \tilde{f}(a) = f(b_1)h\, f(b_2)$ is a well defined mapping of A into H. It remains to check that $\tilde{f}$ is multiplicative. But if $a_i = b_ipb_i'(i = 1, 2)$ and $a = a_1a_2 \in A$, then $b_1'b_2 \in Cv_j\tilde{C}$ for some $j \in J'$. It is straightforward to check using (I) and (T) that under these conditions, $\tilde{f}(a) = \tilde{f}(a_1)\tilde{f}(a_2)$. The theorem now follows from the result of Weil quoted above.

We now consider a family of examples for which the hypotheses of Theorem 1 are satisfied. Let G be a connected semi-simple Lie group and $B = MAN$, a minimal parabolic subgroup (see section 5 for definitions). If $q \in W$ is the element of the Weyl group mapping all positive roots to negative roots, then $G - BqB$ is a finite union of manifolds of dimension less than the dimension of G. Hence $L = BqB$ is large in G. It is easy to check that in this situation $C = \tilde{C} = MA$. However in general, the index set J' of Theorem 1 will be infinite and hence the governing list which results will be quite complicated. If $\tilde{B}$ is any parabolic subgroup, i.e., if $\tilde{B}$ is a subgroup containing a conjugate of B, then for some $\tilde{p} \in G$, $\tilde{L} = \tilde{B}\tilde{p}\tilde{B}$ will be large in G. We now describe several cases where the resulting triple conditions are finite in number.

Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. For $n \geqq 1$, set

$$p_1 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \text{ and } p_2 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. For any matrix g, let ${}^t g$ (respectively, ${}^* g$) denote the matrix transpose (respectively, matrix adjoint) of g. Now set

$$\mathrm{Sp}(n, \mathbf{F}) = \{g \in \mathrm{GL}(2n, \mathbf{F}) : {}^t g p_1 g = p_1\},$$
$$\mathrm{U}(n, n) = \{g \in \mathrm{GL}(2n, \mathbf{C}) : {}^* g p_1 g = p_1\},$$

and for n even,

$$\mathrm{SO}(n, n, \mathbf{F}) = \{g \in \mathrm{SL}(2n, \mathbf{F}) : {}^t g p_2 g = p_2\}.$$

In the following discussion, G will denote one of the groups $\mathrm{GL}(2n, \mathbf{F})$, $\mathrm{SL}(2n, \mathbf{F})$, $\mathrm{Sp}(n, \mathbf{F})$, $\mathrm{U}(n, n)$, $\mathrm{SU}(n, n) = \mathrm{U}(n, n) \cap \mathrm{SL}(2n, \mathbf{C})$, or $\mathrm{SO}(n, n, \mathbf{F})$. Then G is a connected reductive Lie group. Denote the elements of G in block form, e.g.,

$$g = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ for } x, y, z, w \text{ in } \mathbf{F}^{n \times n}.$$

Set

$$C_0 = \left\{c(x, w) = \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix} : x, w \in \mathrm{GL}(n, \mathbf{F})\right\},$$

$$V_0 = \left\{v(z) = \begin{bmatrix} I_n & 0 \\ z & I_n \end{bmatrix} : z \in \mathbf{F}^{n \times n}\right\}$$

$$C = C_0 \cap G,$$

$$V = V_0 \cap G,$$

and

$$B = CV.$$

Then B is a semi-direct product of C and V. For $p = p_2$ in the case $G = \mathrm{SO}(n, n, \mathbf{F})$ and $p = p_1$ in the other cases, $L = BpB = BpV$ is large in G. Indeed, it is routine to check that $\dim_{\mathbf{R}} L = \dim_{\mathbf{R}} G$. If B_0 is a minimal parabolic subgroup of G contained in B and $L_0 = B_0 q B_0$ is the large set described above, it follows that $L \supset L_0$ and hence that L is large in G. Trivially, $C = B \cap p^{-1}Bp = pCp^{-1}$. The decomposition of B into (C, C) double cosets is described by the orbits of the action $c(x, w) \cdot v(z) = v(w^{-1}zx)$ of C on V. By an easy matrix calculation, if $pv(z)p \in L$, then $\det z \neq 0$. A double coset $Cv(z)C$ will be called *non-singular* if $\det z \neq 0$. From linear algebra, there are only finitely many non-singular double cosets. Indeed, for $G = \mathrm{GL}(2n, \mathbf{F})$ or $\mathrm{SL}(2n, \mathbf{F})$ this is trivial. For $G = \mathrm{Sp}(n, \mathbf{R})$, $v(z) \in V$ if and only if $z = {}^tz \in \mathbf{R}^{n\times n}$ and $c(x, w) \in C$ if and only if $x \in \mathrm{GL}(n, \mathbf{R})$ and $w = {}^tx^{-1}$. The action of C on V corresponds to the change of basis of a symmetric bilinear form on $\mathbf{R}^n$. Set

$$z_j = \begin{bmatrix} -I_j & 0 \\ 0 & I_{n-j} \end{bmatrix} \text{ for } j = 0, 1, \ldots, n.$$

Then the double cosets $Cv(z_j)C$ $(j = 0, 1, \ldots, n)$ exhaust the non-singular double cosets of B. The other groups may be treated similarly.

From Theorem 1 it follows that a homomorphism f of B into H extends to a homomorphism $\tilde{f}$ of G into H with $\tilde{f}(p) = h$ if and only if h satisfies the intertwining condition (I) on C and triple conditions (T) for the representatives of the non-singular double cosets of B.

3. Structure of Tits systems with finite Weyl group.

A quadruplet (G, B, M', S) is said to be a *Tits system* if G is a group, B and M' are subgroups of G, S is a subset of $M'/B \cap M'$, and the following axioms are satisfied:

(T1) G is generated by $B \cup M'$ and $M = B \cap M'$ is a normal subgroup of M';

(T2) S is a set of involutive generators of $W = M'/M$;

(T3) $sBs \neq B$ for all $s \in S$;

(T4) $sBp \subset BsB \cup BspB$ for all $(s, p) \in S \times W$.

If $X_1, X_2, \ldots, X_k$ are subsets of W and $B_1, B_2, \ldots, B_{k+1}$ are subsets of B, expressions of the form $B_1X_1B_2X_2 \ldots B_kX_kB_{k+1}$ are understood to mean the set of all quantities $b_1m_1'b_2m_2' \ldots b_km_k'b_{k+1}$ where $b_j \in B_j$ $(j = 1, 2, \ldots, k+1)$, $m_j' \in M'$ and $m_j'M \in X_j$ $(j = 1, 2, \ldots, k)$.

We now summarize some of the consequences of these axioms. For proofs of the results quoted, see Bourbaki [**1**, pp. 9–36]. For s and s' in S, let $n(s, s')$ denote the order of ss' in W (in general, $n(s, s')$ may be infinite). Then S and the set of relations $(ss')^{n(s,s')} = 1$ for $n(s, s') < \infty$ form a presentation of the group W. For $p \in W$, let $l(p)$ denote the length of p, i.e., the smallest integer k for which p may be written as a product of k elements of S. Then (T3) is equivalent to

$$\text{(T3}'\text{)} \quad BsBpB = \begin{cases} BspB \text{ if } l(sp) > l(p) \\ BpB \cup BspB \text{ if } l(sp) < l(p) \end{cases}$$

for all $(s, p) \in S \times W$.

For p and r distinct elements of W, BpB and BrB are disjoint. Moreover, $G = \bigcup_{p \in W} BpB$. Hence the Weyl group W may be identified with $B \backslash G/B$.

We shall henceforth assume that (G, B, M', S) is a Tits system and that W is finite. It then follows from an exercise in Bourbaki [**1**, p. 43] that there is a unique element $q \in W$ with maximal length. Indeed, q may be characterized as the unique element in W for which $l(qs) < l(q)$ for all $s \in S$. It is easy to see that $q^2 = 1$ and $l(pq) = l(qp) = l(q) - l(p)$ for all $p \in W$. In particular this implies that $qSq = S$. Set $\tilde{B} = qBq$.

Lemma 1. *$(G, \tilde{B}, M', S)$ is a Tits system.*

Proof. Since $\tilde{B} \cap M' = B \cap M'$, axioms (T1) and (T2) are satisfied. Axioms (T3) and (T4) are consequences of the fact that $p \rightarrow qpq$ is an automorphism of W which leaves S invariant.

We shall often refer to $(G, \tilde{B}, M', S)$ as the *dual* system.

Lemma 2. *If $(s, p) \in S \times W$ with $l(sp) < l(p)$, then $sBs \cap sp\,B(sp)^{-1} \subset B$.*

Proof. Suppose $l(sp) < l(p)$. Then $l(ssp) > l(sp)$ and from $(T3')$

$$\begin{aligned}(sp)B(sp)^{-1} \cap BsB &= (spB \cap BsBsp)(sp)^{-1}\\ &\subset (BspB \cap BpB)(sp)^{-1}\\ &= \emptyset\end{aligned}$$

The lemma now follows from the fact that $sBs \subset B \cup BsB$.

Let $C = B \cap \tilde{B}$. Lemma 3 shows that C is the subgroup of B normalized by M'.

Lemma 3. $C = \bigcap_{p \in W} pBp^{-1}$.

Proof. Let $s \in S$. Then $sCs \subset B \cup BsB$. But $BsB \cap sCs \subset BsB \cap sqBqs = \emptyset$ by Lemma 2. Hence $sCs \subset B$. By applying the same argument to the dual system, it follows that $sCs \subset \tilde{B}$ and hence that $sCs \subset C$. But then $sCs = C$ and since W is generated by S, $C = pCp^{-1} \subset pBp^{-1}$ for all $p \in W$. Let $C' = \bigcap_{p \in W} pBp^{-1}$. It has just been shown that $C \subset C'$. Since the converse statement is trivial, the lemma is proved.

Now for any $p \in W$, set $N_p = B \cap p^{-1}Bp$ and $U_p = B \cap p^{-1}\tilde{B}p$.

Lemma 4. *The following equations hold for all p and r in W.*

(1) $pN_pp^{-1} = N_{p^{-1}}$.

(2) $N_{qp} = U_p$.

(3) $p^{-1}(N_{p^{-1}} \cap N_r)p = N_p \cap N_{rp} = N_p \cap p^{-1}N_rp$.

(4) $N_p \cap U_p = C$.

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Proof. Statements (1)–(3) follow immediately from the definitions. To prove (4), use Lemma 3 to get

$$N_p \cap U_p = B \cap p^{-1}(B \cap \tilde{B})p = B \cap p^{-1}Cp = B \cap C = C.$$

LEMMA 5. *Let* $(s, p) \in S \times W$ *with* $l(sp) > l(p)$. *Then*
(1) $N_{(sp)^{-1}} \subset N_s$.
(2) $N_{sp} \subset N_p$.
(3) $U_s \subset N_{p^{-1}}$.
(4) $U_p \subset U_{sp}$.

Proof. Let $l(sp) > l(p)$. From $(T3')$, it follows as in Lemma 2 that $BsB \cap pBp^{-1} = \emptyset$ and hence that $sBs \cap pBp^{-1} \subset B$. But then

$$N_{(sp)^{-1}} = B \cap (sp)B(sp)^{-1} \subset sBs \cap B = N_s$$

so (1) is proved. From Lemma 4, we get

$$N_{sp} = (sp)^{-1}N_{(sp)^{-1}}\, sp = (sp)^{-1}\,(N_{(sp)^{-1}} \cap N_s)sp = N_{sp} \cap N_p$$

which proves (2).

To prove (3), it suffices to show that $p^{-1}\, U_s\, p \subset B$ whenever $l(sp) > l(p)$. We do this by induction on the length of p. The case $p = 1$ (i.e., $l(p) = 0$) is trivial. Suppose (3) holds for $p \in W$ and let $p' = ps'$ with $l(sp') > l(p') > l(p)$. Then $l(sp) > l(p)$ and by hypothesis $p^{-1}U_s\, p \subset B$. Hence $(p')^{-1}\, U_s\, p' \subset s'Bs'$. But by Lemma 2

$$(p')^{-1}\, U_s p' = s'Bs' \cap (p')^{-1}\, U_s p' \subset s'Bs' \cap (qsp')^{-1}Bqsp' \subset B$$

so (3) holds for p' as well. By induction (3) holds in general.

(4) follows from (2) by Lemma 4. Indeed, if $\tilde{s} = qsq$, then $l(qp) = l(\tilde{s}qsp) > l(qsp)$ and hence

$$U_p = N_{qp} = N_{\tilde{s}qsp} \subset N_{qsp} = U_{sp}.$$

THEOREM 2. (1) $B = N_pU_p$ *for all* $p \in W$.
(2) *If* $l(sp) > l(p)$ *then* $N_{sp} = N_p \cap p^{-1}N_sp$ *and* $U_{sp} = (p^{-1}U_sp)\cdot\, U_p$.

Proof. The proof will again use induction on the length of p. First consider the case $p = s \in S$. From $(T3')$ it follows that $sBsq \subset BqB$ and hence that $B \subset (sBs)\,(s\tilde{B}s)$. Thus for each element $b \in B$ there exist elements $n \in sBs$ and $u \in s\tilde{B}s$ such that $b = nu$. We claim that $n \in B$. Indeed, if $n \notin B$, then $n \in BsB$. But then we would have $b \in (BsB\; sqB)qs \subset (BqB)qs$ which is impossible since $BsqB \cap BsB = \emptyset$. Hence $n \in B \cap sBs = N_s$ and $u = bn^{-1} \in B \cap s\tilde{B}s = U_s$. Thus we have the decomposition $B = N_sU_s$ for all $s \in S$.

Now suppose the decomposition $B = N_pU_p$ holds for some $p \in W$. Let $s \in S$ with $l(sp) > l(p)$. Since $B = N_sU_s$, then in particular for $n \in N_{p^{-1}}$, there exists $n_1 \in N_s$ and $u_1 \in U_s$ such that $n = n_1u_1$. From Lemma 5, $U_s \subset N_{p^{-1}}$

so $n_1 \in N_s \cap N_{p^{-1}}$. Thus $N_{p^{-1}} = (N_{p^{-1}} \cap N_s) \cdot U_s$. From Lemma 4 it follows that

$$\begin{aligned} B &= N_p U_p \\ &= p^{-1} N_{p^{-1}} p U_p \\ &= (p^{-1}(N_{p^{-1}} \cap N_s)p)\ (p^{-1}U_s p) U_p \\ &= (N_p \cap p^{-1}N_s p)\ (p^{-1}U_s p) U_p. \end{aligned}$$

But by Lemmas 4 and 5, $N_{sp} \subset N_p$ and $N_p \cap p^{-1}N_s p = N_p \cap N_{sp} = N_{sp}$. Also from Lemma 5, $U_p \subset U_{sp}$ and $U_s \subset N_{p^{-1}}$ so

$$p^{-1}U_s p = p^{-1}(U_s \cap N_{p^{-1}})p = U_{sp} \cap N_p.$$

From the argument given above, it follows that

$$U_{sp} = (U_{sp} \cap N_p)\ (U_{sp} \cap U_p) = (p^{-1}U_s p) \cdot U_p.$$

By induction on the length of p, the decomposition (1) holds for all $p \in W$ and the formulae in (2) have been proved as well.

Statements analogous to those in Lemma 5 and part (2) of Theorem 2 hold for the cases $l(ps) < l(p)$, $l(sp) > l(p)$ and $l(sp) < l(p)$. Such statements are easily proved from Lemmas 4 and 5 by substitution of p for p^{-1} and the use of Lemma 5 in the dual system.

Borrowing terminology from the Lie group context, the subgroups $p^{-1}U_s p$ may be called *positive* (respectively, *negative*) *root subgroups* for $(s, p) \in S \times P$ and $l(sp) > l(p)$ (respectively, $l(sp) < l(p)$). It follows easily from Theorem 2 that B is generated by the positive root subgroups. Since the conjugate of a positive root subgroup under q is a negative root subgroup, it follows that G is generated by the positive root subgroups and any element $\bar{q} \in G$ with $\bar{q} \in q$. These statements are of course well known in the context of semi-simple Lie groups. The standard proofs exploit the differentiable structure of such groups. Once the decomposition theorem $B = N_p U_p$ has been proven, the key axiom (T3) for Tits system is an immediate consequence.

4. Governing lists for Tits systems. Let (G, B, M', S) be a Tits system with finite Weyl group. For each $s \in S$ pick once and for all an element $\bar{s} \in M'$ such that $s = \bar{s}M$. Let $\bar{S} = \{\bar{s} : s \in S\}$. Recall that $n(s, s')$ denotes the order of ss' in W. Hence for all $(\bar{s}, \bar{s}') \in \bar{S} \times \bar{S}$, there exists an element $m(\bar{s}, \bar{s}') \in M$ such that $(\bar{s}\,\bar{s}')^{n(s,s')} = m(\bar{s}, \bar{s}')$. Let $(L; Y)$ be any presentation of M. For $\bar{s}$ and $\bar{s}'$ in $\bar{S}$ and $m \in W$, $m(\bar{s}, \bar{s}')$ and $\bar{s}m\bar{s}^{-1}$ may be regarded as words in the elements of the generating set W. Let X_1 be the set of relations of the form $(\bar{s}\,\bar{s}')^{n(s,s')} = m(\bar{s}, \bar{s}')$ and X_2 the set of relations of the form $\bar{s}m\bar{s}^{-1}(\bar{s}m\bar{s}^{-1})^{-1} = 1$. Set $Z = Y \cup X_1 \cup X_2$. Then Z is a set of relations on the generating set $L \cup \bar{S}$ of M'.

LEMMA 7. *$(L \cup \bar{S}\,; Z)$ is a presentation of M'.*

Proof. The lemma follows easily from the fact that $(S; X)$ is a presentation of W where X is the set of relations of the form $(ss')^{n(s,s')} = 1$.

Now for $s \in S$ and U_s as in section 3, consider the double coset space $C\backslash U_s/C$. Let D_s be a set of representatives for the non-trivial double cosets. A typical element of D_s will be denoted by $\bar{u}$. From Lemma 3, $sD_s s \subset BsB$. Hence for each $\bar{u} \in D_s$, we may select elements $v_1(\bar{u}) \in B$ and $v_2(\bar{u}) \in U_s$ such that $\bar{s}\bar{u}\bar{s} = v_1(\bar{u})\,\bar{s}\,v_2(\bar{u})$. Let $(B; Z_1)$ be the group-table presentation of B. Let Z_2 be the set of relations of the form $\bar{s}n\bar{s}^{-1}\,\tilde{n} = 1$ for $\bar{s} \in \bar{S}$, $u \in N_s$ and $\tilde{n} = (\bar{s}n\bar{s}^{-1})^{-1}$ in B. Let Z_3 be the set of relations of the form

$$\bar{s}\bar{u}\bar{s}(v_1(\bar{u})\bar{s}v_2(\bar{u}))^{-1} = 1$$

for $\bar{s} \in \bar{S}$ and $\bar{u} \in D_s$. For Z as above, set $\tilde{Z} = Z \cup Z_1 \cup Z_2 \cup Z_3$. Then $\tilde{Z}$ may be regarded as a set of relations on the generating set $B \cup \bar{S}$ of G.

THEOREM 3. *Let* $f\colon B \to H$ *be a homomorphism. Select a set of elements* $H_{\bar{s}} = \{h_{\bar{s}} : \bar{s} \in \bar{S}\}$. *Then* f *extends to a homomorphism* $\tilde{f}\colon G \to H$ *with* $\tilde{f}(s) = h_{\bar{s}}$ *if and only if the following conditions are satisfied*:

(W) $(h_{\bar{s}}h_{\bar{s}'})^{n(s,s')} = m(\bar{s}, \bar{s}')$ for $(\bar{s}, \bar{s}) \in \bar{S} \times \bar{S}$.

(I) $h_{\bar{s}}f(n) = f(\bar{s}\,n\,\bar{s}^{-1})h_{\bar{s}}$ for $(\bar{s}, n) \in \bar{S} \times N_s$.
(T) $h_{\bar{s}}f(\bar{u})h_{\bar{s}} = f(v_1(\bar{u}))h_{\bar{s}}f(v_2(\bar{u}))$ for $(\bar{s}, \bar{u}) \in \bar{S} \times D_s$.

Proof. The statement of the theorem is equivalent to the statement that $(\bar{S} \cup B; \tilde{Z})$ is a presentation of G. Let $\mathscr{G}$ be the free group on the generating set $\bar{S} \cup B$. The set of relations $\tilde{Z}$ may be identified with a set of words in $\mathscr{G}$ which describe the identity element in G. By an elementary word operation we shall mean insertion or deletion of any element from $\tilde{Z}$. By a word operation on $\mathscr{G}$, we shall mean a composite of elementary word operations. In order to show that $(\bar{S} \cup B; \tilde{Z})$ is a presentation of G, it suffices to show that whenever w is a word in $\mathscr{G}$ which represents the identity element in G, then w may be mapped to the identity word by a word operation.

By Lemma 7, two words in the elements of $\bar{S} \cup M$ which represent the same group element in M' may be mapped into one another by a word operation. Hence every element in $\mathscr{G}$ may be regarded as a word of the form $b_1m_1b_2m_2 \ldots b_nm_n$ where $b_j \in B$ and $m_j \in M'$ for $j = 1, 2, \ldots, k$. To prove the theorem it suffices to show that every word of the form m_1bm_2 may be mapped to a word of the form $b_1m_3b_2$ by a word operation. Let $p = m_1M$ and $p' = m_2M$. By induction on the length of p, it suffices to consider the case $m_1 = \bar{s} \in \bar{S}$. Since $sBp' = N_s sU_s p'$, we may assume $b = u \in U_s - C$. There are now two cases. First assume $l(sp') > l(p') = k$. Write p' in the form $p' = s_1s_2 \ldots s_k$. Let $p_0 = 1$ and $p_j = s_1s_2 \ldots s_j$ for $j = 1, 2, \ldots, k$. Then for all j, $l(sp_j) > l(p_j)$ and by Lemma 5, $U_s \subset N_{p_j^{-1}}$. It follows that $p_{j-1}{}^{-1} U_s p_{j-1} \subset N_{s_j}$ for $j = 1, 2, \ldots, k$. Set $m_2 = \bar{s}_1\bar{s}_2 \ldots \bar{s}_k m_0$ for some $m_0 \in M$, $u_0 = u$, $u_j = (\bar{s}_1\bar{s}_2 \ldots \bar{s}_j)^{-1}\,u(\bar{s}_1\bar{s}_2 \ldots \bar{s}_j)^{-1}$ for $j = 1, 2, \ldots, k$, and $b_2 = u_k m_0$. Then $\bar{s}um_2$ may be mapped to $\bar{s}m_2b_2$ by successively replacing $u_j\bar{s}_{j+1}$ by $\bar{s}_{j+1}u_{j+1}$ for $j = 0, 1, 2, \ldots, k - 1$. Clearly such replacements may be described by elementary word operations. Now consider the case $l(sp') < l(p')$. Then $\bar{s}u$ may be mapped by a word operation to a word of the form $v_1\bar{s}v_2(\bar{s})^{-1}$ for $v_1 \in B$ and $v_2 \in U_s$. Since $l(ssp') > l(sp')$, the argument just given implies that $v_2(\bar{s})^{-1}m_2$ may be mapped by a word operation to a word of the form $\bar{s}^{-1}m_2b_2$. This completes the proof of the theorem.

5. Examples.

Example 1. Let $G = \mathrm{GL}(n, k)$ for $n \geqq 2$ and k any field. Let B be the upper triangular subgroup of G and M' the set of elements in G having precisely one non-zero entry in each row and column. Then $M = B \cap M'$ is the set of diagonal elements in G and $W = M'/M$ may be identified with the permutation group on n letters. For $i = 1, 2, \ldots, n - 1$ let s_i be the element of W identified with the transposition $(i, i + 1)$. Then (G, B, M', S) is a Tits system [**1**, p. 24]. The element $q \in W$ of maximal length is the coset determined by any matrix in M' with non-zero elements along the secondary diagonal. For $1 \leqq i \neq j \leqq n - 1$, s_is_j is of order 2 if $|i - j| > 1$ and of order 3 if $|i - j| = 1$. $\tilde{B} = qBq$ is the lower triangular subgroup and $B \cap \tilde{B} = M$. For $i = 1, 2, \ldots, n - 1$,

$$N_{s_i} = \{b \in B : b_{i,i+1} = 0\}$$

and

$$U_{s_i} = \{b \in B\colon b_{kl} = 0 \text{ whenever } k \neq l \text{ and } (k, l) \neq (i, i + 1)\}.$$

U_{s_i} contains precisely one non-trivial (M, M) double coset. Hence in the governing list for this system given by Theorem 3, there are $n - 1$ triple conditions.

Now let $G_0 = \mathrm{SL}(n, k)$ $B_0 = G_0 \cap B$ and $M_0' = M' \cap G_0$. Then (G_0, B_0, M_0', S) is again a Tits system. For $n \geqq 3$, the resulting governing list again contains $n - 1$ triple conditions. For $n = 2$, however, there are $[k^*\colon (k^*)^2]$ such conditions. Here $k^* = \{x \in k\colon x \neq 0\}$, $(k^*)^2 = \{x^2\colon x \in k^*\}$ and $[k^*\colon (k^*)^2]$ is the cardinality of $k^*/(k^*)^2$.

Example 2. Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The notation follows that of Kunze and Stein [**2**, pp. 385–392]. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and Δ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$. Fix $H_0 \in \mathfrak{a}$ with $\alpha(H_0) \neq 0$ for all $\alpha \in \Delta$ and let Δ^+ (respectively, Δ^-) be the set of $\alpha \in \Delta$ for which $\alpha(H_0) > 0$ (respectively, $\alpha(H_0) < 0$). Set $\mathfrak{n} = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ and $\mathfrak{v} = \sum_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$. Let K, A, N, and V denote the Lie sugbroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{v}$. Let M be the centralizer of A in K, M' the normalizer of A in K and $W = M'/M$. Via the contragredient of the adjoint representation, W may be regarded as a transformation group on Δ. Select a set $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of simple roots in Δ^+. For $\alpha_i \in \Pi$ let s_i denote the reflection of Δ in α_i. Then $S = \{s_1, s_2, \ldots, s_n\}$ is a set of involutive generators for W. Set $B = MAN$. Then (G, B, M', S) is a Tits system. The element $q \in W$ having maximal length in the elements of S is the unique element in W mapping Δ^+ onto Δ^-. It follows that $\tilde{B} = qBq = MAV$ and $B \cap \tilde{B} = MA$. Also

$$U_{s_i} = MA \exp(\mathfrak{g}_{\alpha_i} + \mathfrak{g}_{2\alpha_i}) \quad \text{and} \quad N_{s_i} = MA \exp\Big(\sum_{\alpha \in \Delta^+ - \{\alpha_i, 2\alpha_i\}} \mathfrak{g}_\alpha\Big).$$

For $i \neq j$, the order $n(s_i, s_j)$ of $s_i s_j$ is either 2, 3, 4, or 6.

Suppose, in particular, that G has a complex structure. Then U_{s_i}/MA is a one parameter subgroup for $i = 1, 2, \ldots, n$. Thus the governing list of Theorem 3 contains precisely one triple condition for each simple root.

Now return to the general situation. Let T be a strongly continuous representation of B on a Banach space $\mathfrak{B}$. Suppose that T' is an extension of T to an algebraic homomorphism of G into $\mathrm{GL}(B)$. It is easy to see that T' is strongly continuous. If $\mathfrak{B}$ is a Hilbert space, then T' is uniformly bounded if and only if T is uniformly bounded. In the complex case, B is solvable and it follows that any uniformly bounded representation of B on an infinite dimensional Hilbert space is similar to a unitary representation. Thus, up to similarity, all infinite dimensional uniformly bounded representations of G arise from unitary representations of B via the governing lists of Theorem 3.

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AN L^p SATURATION THEOREM FOR SPLINES

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1. Let $\Delta_n : 0 = x_0^{(n)} < x_1^{(n)} < \ldots < x_n^{(n)} = 1$ be a subdivision of $[0, 1]$, and let $\mathscr{S}_k(\Delta_n)$ denote the class of functions whose restriction to each sub-interval $[x_{i-1}^{(n)}, x_i^{(n)})$ is a polynomial of degree at most k. Gaier [**1**] has shown that for uniform subdivisions Δ_n (that is, subdivisions for which $x_i^{(n)} = i/n$)

$$||f - \mathscr{S}_k(\Delta_n)||_p = o(n^{-k-1})$$

if and only if f is a polynomial of degree at most k. Here, and subsequently, $||\cdot||_p$ denotes the usual norm in $L^p[0, 1]$, $1 \leqq p \leqq \infty$, and we should emphasize that functions differing only on a set of Lebesgue measure zero are identified.

One of the authors [**4**] has recently characterized those functions f for which

$$||f - \mathscr{S}_k(\Delta_n)||_\infty = O(n^{-k-1}).$$

In this paper we solve the corresponding problem for the L^p norms, $1 \leqq p < \infty$. Let

$$\text{Lip}(1, L^p) = \left\{f : \left(\int_0^1 |f(x + \delta) - f(x)|^p dx\right)^{1/p} = O(\delta)\right\}$$

(f assumed to be identically zero outside $[0, 1]$) and define

$$\mathscr{L}_p^{\,k} = \{f : f \in C^{k-1}[0, 1], f^{(k-1)} \text{ is absolutely continuous}, f^{(k)} \in \text{Lip}(1, L^p)\}.$$

Our main result is the following

THEOREM. *Let f be a real-valued function on* $[0, 1]$ *and let* $\{\Delta_n\}_{n=1}^\infty$ *be uniform subdivisions. Then*

$$||f - \mathscr{S}_k(\Delta_n)||_p = O(n^{-k-1}), \tag{1}$$

if and only if $f \in \mathscr{L}_p^{\,k}$.

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2. In this section we shall demonstrate the sufficiency part of (1); in fact we shall establish the following more general result, namely,

LEMMA 3. *Let $f \in \mathscr{L}_p^{\,k}$. Then, given any sequence of arbitrary subdivisions $\{\Delta_n\}_{n=1}^\infty$, there exists a sequence of spline functions $\{S_n\}_{n=1}^\infty$ of degree k with knots at the points of Δ_n* (i.e., $S_n \in \mathscr{S}_k(\Delta_n) \cap C^{k-1}[0, 1]$) *satisfying*

$$||f - S_n||_p = O(||\Delta_n||^{k+1}), \tag{2}$$

where $||\Delta_n|| = \max_{1 \leqq i \leqq n} (x_i^{(n)} - x_{i-1}^{(n)})$.

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We must first prove two other lemmas.

LEMMA 1. *If* $f \in \mathrm{Lip}(1, L^p)$, *then*

$$||f - \mathcal{S}_0(\Delta_n)||_p = O(||\Delta_n||) \tag{3}$$

Proof. We first remark that $f \in \mathrm{Lip}(1, L^p)$ implies via the Hölder inequality that $f \in \mathrm{Lip}(1, L^1)$, so that f is of bounded variation [**2**] and hence $f \in L^\infty[0, 1]$. Let us define

$$D_n = \int_0^{||\Delta_n||} \sum_{i=0}^{n-1} \int_0^{x_{i+1}-x_i} |f(t + x_i) - f(x + x_i)|^p \, dx \, dt.$$

With the change of variables

$$u = x + x_i, \qquad v = t - x,$$

and since $f \in \mathrm{Lip}(1, L^p)$, we have

$$\begin{aligned} D_n &\leqq \sum_{i=0}^{n-1} \int_{-||\Delta_n||}^{||\Delta_n||} \int_{x_i}^{x_{i+1}} |f(u + v) - f(u)|^p \, du \, dv \\ &\leqq K||\Delta_n||^{p+1}, \end{aligned}$$

where K is independent of n. Thus there exists τ_n, $0 \leqq \tau_n \leqq ||\Delta_n||$, such that

$$\sum_{i=0}^{n-1} \int_0^{x_{i+1}-x_i} |f(\tau_n + x_i) - f(x + x_i)|^p \, dx \leqq K||\Delta_n||^p.$$

Choosing σ_n to be the step function taking the value $f(\tau_n + x_i)$ on $[x_i, x_{i+1})$, it follows that

$$||f - \sigma_n||_p = O(||\Delta_n||).$$

LEMMA 2. *Let* $f \in L^p[0, 1]$ *and* ν *be a natural number. If* $s_n \in \mathcal{S}_{\nu-1}(\Delta_n) \cap C^{\nu-2}$, $n = 1, 2, \ldots,$ *and* ϕ *is a positive function on the natural numbers such that*

$$||f - s_n||_p = O(\phi(n)), \tag{4}$$

then there exists a sequence $\{S_n\}_{n=1}^{\infty}$, $S_n \in \mathcal{S}_\nu(\Delta_n) \cap C^{\nu-1}$, *satisfying*

$$||F - S_n||_p = O(||\Delta_n||\phi(n)) \tag{5}$$

where

$$F(x) = \int_0^x f(t)dt.$$

(C^{-1} *is interpreted to be the space of all real-valued functions.*)

Proof. Consider the B-spline $M_i(x)$ of degree $\nu - 1$ for the subdivision Δ_n, $i = 0, 1, \ldots, n - \nu$, defined in [**6**]. It is known [**6**] that $M_i(x)$ is a non-negative function having support in $[x_i, x_{i+\nu}]$, $M_i \in \mathcal{S}_{\nu-1} \cap C^{\nu-2}$ and

$$\int_0^1 M_i(x)dx = 1. \tag{6}$$

Following [3], define

$$A_i = \int_{x_i}^{x_{i+1}} (f(t) - s_n(t))dt, \qquad i = 0, 1, \ldots, n-1,$$

$$\tilde{A}_i = \int_{x_i}^{x_{i+1}} |f(t) - s_n(t)|dt, \qquad i = 0, 1, \ldots, n-1,$$

and

$$S_n(x) = \int_0^x s_n(t)dt + \sum_{i=0}^{n-\nu} A_i \int_0^x M_i(t)dt.$$

Suppose that $x_i \leqq x \leqq x_{i+1}$. Then

$$\begin{aligned} F(x) - S_n(x) &= \int_0^x (f(t) - s_n(t))dt - \sum_{j=0}^{n-\nu} A_j \int_0^x M_j(t)dt \\ &= \int_{x_i}^x (f(t) - s_n(t))dt + \sum_{j=0}^{i-1} A_j - \sum_{j=0}^{n-\nu} A_j \int_0^x M_j(t)dt. \end{aligned}$$

By (6),

$$\int_0^x M_j(t)dt = 1 \quad \text{for} \quad j \leqq i - \nu \quad \text{and} \quad \int_0^x M_j(t)dt = 0 \quad \text{for} \quad j \geqq i + 1.$$

Thus

$$\begin{aligned} |F(x) - S_n(x)| &\leqq \left| \int_{x_i}^x (f(t) - s_n(t))dt \right| + \left| \sum_{j=i-\nu+1}^{i-1} A_j \left(1 - \int_0^x M_j(t)dt\right) \right| \\ &\qquad + \left| A_i \int_0^x M_i(t)dt \right| \\ &\leqq \tilde{A}_i + \sum_{j=i-\nu+1}^{i} |A_j| \leqq K \left[\sum_{j=i-\nu+1}^{i} \tilde{A}_j^p \right]^{1/p} \end{aligned}$$

for some constant $K = K(\nu)$. Hence

$$\int_{x_i}^{x_{i+1}} |F(x) - S_n(x)|^p dx \leqq K^p \sum_{j=i-\nu+1}^{i} \tilde{A}_j^p (x_{i+1} - x_i). \tag{7}$$

But by Hölder's inequality,

$$\begin{aligned} \tilde{A}_j^p &\leqq (x_{j+1} - x_j)^{p-1} \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt \\ &\leqq ||\Delta_n||^{p-1} \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt; \end{aligned}$$

thus

$$\begin{aligned} &\int_0^1 |F(x) - S_n(x)|^p dx \\ &\qquad \leqq K^p ||\Delta_n||^{p-1} \sum_{i=0}^{n-1} \sum_{j=i-\nu+1}^{i} (x_{i+1} - x_i) \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt \end{aligned}$$

and by reversing the order of summation, we obtain

$$\int_0^1 |F(x) - S_n(x)|^p dx \leqq K^p ||\Delta_n||^{p-1} \sum_{j=0}^{n-1} (x_{j+\nu} - x_j) \int_{x_j}^{x_{j+1}} |f(t) - s_n(t)|^p dt \tag{8}$$

$$\leqq K_1^p ||\Delta_n||^p \int_0^1 |f(t) - s_n(t)|^p dt$$

for some constant $K_1 = K_1(\nu)$. The lemma follows on applying (4) in (8).

Lemma 3 is an easy consequence of Lemmas 1 and 2.

3. We now seek to establish the necessity part of (1). In this section, all subdivisions Δ_n are assumed to be uniform.

It will be convenient to state at this point the L^p version of Markoff's inequality and of two inequalities due to Gaier.

LEMMA 4 (Markoff [**1**]). *Let P be a polynomial of degree k on $[a, b]$ and $0 \leqq j \leqq k$. Then there exists a constant $K = K(p, k, j)$ such that*

$$\left[\int_a^b |P^{(j)}(x)|^p dx \right]^{1/p} \leqq K(b - a)^{-j} \left[\int_a^b |P(x)|^p dx \right]^{1/p}. \tag{9}$$

LEMMA 5 (Gaier [**1**]). *Suppose P is a function on $[-a, b]$ which reduces to a polynomial of degree k on each of $[-a, 0]$ and $(0, b]$, and define $h = P(0+) - P(0-)$. Then there exists a constant $K = K(p, k)$ such that*

$$|h| \leqq K(\min (a, b))^{-1/p} \left[\int_{-a}^b |P(x)|^p dx \right]^{1/p}. \tag{10}$$

LEMMA 6 (Gaier [**1**]). *Let $T_j \in \mathscr{S}_k(\Delta_j)$, $j = n, n + 1$, and let $h_i^{(\nu)}$ denote the jumps of $T_n^{(\nu)}$ at $x_i = i/n$, $i = 1, \ldots, n - 1$, i.e., $h_i^{(\nu)} = T_n^{(\nu)}(x_i+) - T_n^{(\nu)}(x_i-)$. Let $0 < \epsilon < 1/2$. Then there exists a constant $C = C(\epsilon)$ such that*

$$|h_i^{(\nu)}| \leqq Cn^{\nu+1/p} \left[\int_{i/(n+1)}^{(i+1)/(n+1)} |T_{n+1} - T_n|^p dx \right]^{1/p} \tag{11}$$

if $\epsilon \leqq i/n \leqq (1 - \epsilon)$.

Next we prove

LEMMA 7. *Let*

$$||f - S_n||_p = O(n^{-k-1}) \tag{12}$$

for some sequence $S_n \in \mathscr{S}_k(\Delta_n)$, $n = 1, 2, \ldots$

Then

$$f \in C^{k-1}[0, 1],\ f^{(k-1)} \text{ is absolutely continuous} \tag{13}$$

and

$$||f^{(k)} - S_{2^n}^{(k)}||_p = O(2^{-n}). \tag{14}$$

Proof. Since $S_{2^n} - S_{2^{n+1}} \in \mathcal{S}_k(\Delta_{2^{n+1}})$, we may apply (9) to obtain

$$\|S_{2^n}^{(\nu)} - S_{2^{n+1}}^{(\nu)}\|_p \leqq K(2^{n+1})^\nu \|S_{2^n} - S_{2^{n+1}}\|_p \leqq K_1 2^{-n(k+1-\nu)}, \qquad \nu = 1, 2, \ldots, k, \tag{15}$$

where $K = K(p, k, \nu)$, $K_1 = K_1(p, k, \nu)$ are constants.

Hence there are functions $f_\nu \in L^p[0, 1]$ satisfying

$$\|S_{2^n}^{(\nu)} - f_\nu\|_p \leqq K_2 2^{-n(k+1-\nu)} \tag{16}$$

for some constant $K_2 = K_2(p, k, \nu)$.

Therefore $S_{2^n}{}^{(\nu)} \to f_\nu$ a.e. on $[0, 1]$ and so for almost all ϵ in $[0, 1]$, we have

$$S_{2^n}^{(\nu-1)}(\epsilon) \to f_{\nu-1}(\epsilon). \tag{17}$$

Let $h_i{}^{(\nu)}$ denote the jump of $S_{2^n}{}^{(\nu)}$ at $x_i = i2^{-n}$, and let

$$x_+{}^0 = \begin{cases} 1, x \geqq 0, \\ 0, x < 0. \end{cases}$$

For $x \in [\epsilon, 1 - \epsilon]$, we define

$$\begin{aligned} P_{2^n}(x) &= \int_\epsilon^x S_{2^n}^{(\nu)}(t)dt \\ &= S_{2^n}^{(\nu-1)}(x) - S_{2^n}^{(\nu-1)}(\epsilon) - \sum_i{}' h_i^{(\nu-1)}[x - i/2^n]_+{}^0, \end{aligned} \tag{18}$$

where $\sum_i{}'$ means that we sum over those i for which $\epsilon \leqq i2^{-n} \leqq 1 - \epsilon$.

By (11),

$$\begin{aligned} \left[\sum_i{}' |h_i^{(\nu-1)}|^p\right]^{1/p} &\leqq C(\epsilon)2^{n(\nu-1+1/p)}\|S_{2^n} - S_{2^{n+1}}\|_p \\ &\leqq C_1(\epsilon)2^{-n(k+2-\nu-1/p)}. \end{aligned} \tag{19}$$

Let

$$\bar{f}(x) = \int_\epsilon^x f_\nu(t)dt + f_{\nu-1}(\epsilon). \tag{20}$$

Then,

$$\begin{aligned} &\left[\int_\epsilon^{1-\epsilon} |f_{\nu-1}(x) - \bar{f}(x)|^p dx\right]^{1/p} \\ &\quad\leqq \|f_{\nu-1} - S_{2^n}^{(\nu-1)}\|_p + \left[\int_\epsilon^{1-\epsilon} |S_{2^n}^{(\nu-1)}(x) - P_{2^n}(x) - f_{\nu-1}(\epsilon)|^p dx\right]^{1/p} \\ &\qquad + \left[\int_0^1 |P_{2^n}(x) + f_{\nu-1}(\epsilon) - \bar{f}(x)|^p dx\right]^{1/p} \\ &\quad= \alpha_1{}^n + \alpha_2{}^n + \alpha_3{}^n, \text{ say.} \end{aligned} \tag{21}$$

From (17)–(19),

$$(22)\quad \alpha_2{}^n \leqq \left[\int_\epsilon^{1-\epsilon} |S_{2^n}^{(\nu-1)}(\epsilon) - f_{\nu-1}(\epsilon)|^p dx\right]^{1/p} + \left[\sum_i{}' |h_i^{(\nu-1)}|^p\right]^{1/p} \rightarrow 0 \text{ as } n \rightarrow \infty .$$

Using (16), (18) and (20), we have

$$(23)\quad \alpha_3{}^n \leqq \left[\int_0^1 \int_\epsilon^x |S_{2^n}^{(\nu)}(t) - f_\nu(t)|^p dt\, dx\right]^{1/p} \leqq ||S_{2^n}^{(\nu)} - f_\nu||_p \rightarrow 0 \text{ as } n \rightarrow \infty .$$

Hence, letting $n \rightarrow \infty$ in (21) and using (16), (22) and (23), we obtain

$$f_{\nu-1}(x) = \bar{f}(x) = \int_\epsilon^x f_\nu(t)dt + f_{\nu-1}(\epsilon)$$

a.e. on $[\epsilon, 1 - \epsilon]$, $\nu = 1, 2, \ldots, k$, for almost all ϵ in $[0, 1]$. Since $f_0 = f$, it follows that $f_\nu(x) = f^{(\nu)}(x)$ a.e. on $[0, 1]$, and this together with (16) establishes (14). If $k = 0$, (13) is redundant. If $k = 1$, we have

$$f(x) = \int_\epsilon^x f_1(t)dt + f(\epsilon) \text{ a.e. on } [\epsilon, 1 - \epsilon]$$

for almost all ϵ in $[0, \frac{1}{2}]$, and from (16), $f_1 \in L^p[0, 1]$, and hence $f_1 \in L^1[0, 1]$.

Thus we may find $x_0 \in (0, 1)$ and a sequence $\epsilon_1 > \epsilon_2 > \ldots \rightarrow 0$ such that

$$(x_0) = \int_{\epsilon_i}^{x_0} f_1(t)dt + f(\epsilon_i).$$

It follows that $f(\epsilon_i) \rightarrow \phi_0$, say, as $i \rightarrow \infty$ and defining

$$\phi(x) = \int_0^x f_1(t)dt + \phi_0,$$

we have $f(x) = \phi(x)$ a.e. on $[0, 1]$, and so f is equivalent to an absolutely continuous function on $[0, 1]$, which is (13).

Essentially similar arguments enable us to establish (13) for general values of k. This completes the proof of the lemma.

LEMMA 8. *Let* ξ_i, $i = 1, 2, \ldots, m$, *be rational numbers with a common denominator* q *such that* $\min(\xi_i, 1 - \xi_i) \leqq 1/4$, $i = 1, 2, \ldots, m$. *Let* h_i, $i = 1, 2, \ldots, m$, *be real numbers such that*

$$(24)\quad \sum_{i=1}^m |h_i|^p q^{p-1} \geqq N > 0,$$

and let n be any integral multiple of q. Then there exists an integer l with $n + 1 \leqq l \leqq 2n$, *such that*

$$\sum_{i=1}^{m} |h_i|^p [\rho(\xi_i, \Delta_l)]^{kp+1} \geqq cNl^{-p(k+1)} \tag{25}$$

for some absolute constant c, where $\rho(\xi_i, \Delta_l) = \inf_{0 \leqq j \leqq l} |\xi_i - j/l|$.

Proof. Assume first that $1/4 \geqq \xi_1 > \xi_2 > \ldots > \xi_m \geqq 1/q$. Define t_i to be $n\xi_i$, $i = 1, 2, \ldots, m$. Then t_i is a natural number no greater than $n/4$. Fix i and let r be a natural number with $1 \leqq r \leqq n$. Then either we have (i) $h/\xi_i \leqq r < (h + \frac{1}{2})/\xi_i$ for some natural number h, or we have (ii) $(h - \frac{1}{2})/\xi_i \leqq r < h/\xi_i$ for some natural number h.

If case (i) applies, then, since $\xi_i = t_i/n$, we have $hn/t_i \leqq r < (h + \frac{1}{2})n/t_i$, and so

$$0 \leqq \frac{t_i}{n} - \frac{t_i + h}{n + r} = \frac{t_i r - hn}{n(n + r)} \leqq \frac{1}{2(n + r)}.$$

Thus

$$\rho(\xi_i, \Delta_{n+r}) = (t_i r - hn)/(n(n + r)) = \xi_i s/(n + r), \text{ where } s = r - hn/t_i \tag{26}$$

If case (ii) applies, we have

$$\left(h - \tfrac{1}{2}\right) \frac{n}{t_i} \leqq r < \frac{hn}{t_i}, \text{ and so}$$

$$0 < \frac{t_i + h}{n + r} - \frac{t_i}{n} = \frac{hn - t_i r}{n(n + r)} \leqq \frac{1}{2(n + r)}.$$

Hence

$$\rho(\xi_i, \Delta_{n+r}) = \frac{hn - t_i r}{n(n + r)} = \frac{\xi_i s}{n + r}, \tag{27}$$

where $s = hn/t_i - r$. Therefore we have

$$\begin{aligned}
\sum_{r=n+1}^{2n} [\rho(\xi_i, \Delta_r)]^{kp+1} &= \sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\
&= \sum_{h=0}^{t_i - 1} \sum_{hn/t_i \leqq r < (h+\frac{1}{2})n/t_i} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\
&\qquad + \sum_{h=1}^{t_i} \sum_{(h-\frac{1}{2})n/t_i \leqq r < hn/t_i} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \\
&\geqq 2 \sum_{h=0}^{t_i - 1} \sum_{s=1}^{[\frac{1}{2}n/t_i]} \left(\frac{\xi_i s}{2n}\right)^{kp+1}
\end{aligned}$$

where we have used (26) and (27) to estimate the sums over r.

Using the integral test to estimate the inner sum above, we obtain

$$\sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \geqq \frac{2t_i}{kp + 2} \left(\frac{\xi_i}{2n}\right)^{kp+1} \left(\frac{n}{2t_i} - 1\right)^{kp+2}.$$

Since $n/2t_i = 1/2\xi_i \geqq 2$, we have $n/2t_i - 1 \geqq n/4t_i$. Hence

$$\sum_{r=1}^{n} [\rho(\xi_i, \Delta_{n+r})]^{kp+1} \geqq (kp+2)^{-1}2^{-3kp-4}t_i\left(\frac{\xi_i}{n}\right)^{kp+1}\left(\frac{n}{t_i}\right)^{kp+2} \tag{28}$$
$$= C_1 n^{-kp}.$$

Thus from (24) and (28),

$$\sum_{r=n+1}^{2n} \sum_{i=1}^{m} |h_i|^p[\rho(\xi_i, \Delta_r)]^{kp+1} = \sum_{i=1}^{m} \sum_{r=n+1}^{2n} |h_i|^p[\rho(\xi_i, \Delta_r)]^{kp+1}$$
$$\geqq C_1 N n^{-kp} q^{1-p}.$$

Hence there exists l with $n + 1 \leqq l \leqq 2n$ such that

$$\sum_{i=1}^{m} |h_i|^p[\rho(\xi_i, \Delta_l)]^{kp+1} \geqq c_1 N n^{-1-kp} q^{1-p}$$
$$\geqq cNl^{-p(k+1)}$$

for some constant c, since $q < l \leqq 2n$. It is obvious that the proof may easily be modified for the slightly more general statement of the lemma.

LEMMA 9. *Suppose*

$$\left[\int_0^1 |f - S_n(t)|^p dt\right]^{1/p} \leqq Kn^{-k-1} \tag{29}$$

for some sequence $S_n \in \mathscr{S}_k(\Delta_n)$, $n = 1, 2, \ldots$. *Then there is a constant* C *such that for* $n = 1, 2, \ldots,$

$$\sum_{i=1}^{n} |h_i^{(k)}|^p n^{p-1} \leqq C,$$

where $h_i^{(k)}$, $i = 1, 2, \ldots, n$ *are the jumps of* $S_n^{(k)}$.

Proof. Applying Lemma 6 with $T_j = S_j, j = n, n+1$, and with $\nu = k$, $\epsilon = 1/4$, we deduce that

$$\sum_i{}' |h_i^{(k)}|^p n^{-1-pk} \leqq K \int_0^1 |S_{n+1} - S_n|^p \leqq K_1 n^{-p(k+1)}$$

for some constants K, K_1. If Lemma 9 is to be false, we may suppose, without loss of generality, that given $N > 0$, there exists a natural number q such that

$$\sum_{i=1}^{m} |h_i^{(k)}| q^{p-1} \geqq N,$$

where $1/4 \geqq \xi_m > \xi_{m-1} > \ldots > \xi_1 = 1/q$ are the points of $[0, 1/4] \cap \Delta_q$ and $h_i^{(k)}$, $i = 1, 2, \ldots, m$, are the corresponding jumps of $S_q^{(k)}$. By Lemma 8 therefore, we can find l with $2q + 1 \leqq l \leqq 4q$ satisfying

$$\sum_{i=1}^{m} |h_i^{(k)}|^p[\rho(\xi_i, \Delta_l)]^{kp+1} \geqq cNl^{-p(k+1)}. \tag{30}$$

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Let $\xi_i \in [r_i/l, (r_i + 1)/l] = I_{r_i}$, where r_i is an integer. Applying Lemmas 4 and 5 to the function $S_q(x) - S_l(x)$ on I_{r_i}, we have

$$\begin{aligned} |h_i^{(k)}| &\leqq K_2[\rho(\xi_i, \Delta_l)]^{-1/p}\left[\int_{I_{r_i}} |S_q^{(k)} - S_l^{(k)}|^p dt\right]^{1/p} \\ &\leqq K_3[\rho(\xi_i, \Delta_l)]^{-k-1/p}\left[\int_{I_{r_i}} |S_q - S_l|^p dt\right]^{1/p} \end{aligned}$$

so that, since the intervals I_{r_i} are pairwise disjoint,

$$\begin{aligned} \int_0^1 |S_q - S_l|^p dt &\geqq K_4 \sum_{i=1}^{m} |h_i^{(k)}|^p[\rho(\xi_i, \Delta_l)]^{kp+1} \\ &\geqq K_5 N l^{-p(k+1)} \end{aligned}$$

by (30). Thus

$$\begin{aligned} \|f - S_l\|_p &\geqq \|S_q - S_l\|_p - \|f - S_q\|_p \\ &\geqq K_5 N l^{-k-1} - K q^{-k-1} \\ &\geqq (K_5 N - 4^{k+1} K) l^{-k-1} \end{aligned}$$

which contradicts (29) if N is sufficiently large. This proves the lemma.

We are now in a position to prove the necessity part of the theorem which we state as

LEMMA 10. *If* $\|f - \mathscr{S}_k\|_p = O(n^{-k-1})$, *then* $f \in \mathscr{L}_k{}^p$.

Proof. Suppose $K > 0$ and $S_n \in \mathscr{S}_k(\Delta_n)$ satisfy

$$\left[\int_0^1 |f - S_n|^p dt\right]^{1/p} \leqq K n^{-k-1}, \qquad n = 1, 2, \ldots . \tag{31}$$

By Lemma 7, $f \in C^{k-1}[0, 1]$, $f^{(k-1)}$ is absolutely continuous on $[0, 1]$, and

$$\left[\int_0^1 |f^{(k)} - \sigma_{2^n}|^p dt\right]^{1/p} \leqq K' 2^{-n}, \qquad n = 1, 2, \ldots \tag{32}$$

where $\sigma_n = S_n{}^{(k)}$. Let $0 < \delta < 1$ and choose n such that $2^{-n-1} \leqq \delta < 2^{-n}$. Let $h_i{}^{(\nu)}, i = 1, 2, \ldots, 2^n - 1$ be the jumps of $S_{2^n}{}^{(\nu)}, \nu = 0, 1, \ldots, k$. If $0 \leqq i \leqq 2^n - 2$, we have

$$\int_{i2^{-n}}^{(i+1)2^{-n}} |\sigma_{2^n}(t + \delta) - \sigma_{2^n}(t)|^p dt \leqq 2^{-n}|h_{i+1}^{(k)}|^p, \tag{33}$$

and

$$\int_{1-2^{-n}}^{1-\delta} |\sigma_{2^n}(t + \delta) - \sigma_{2^n}(t)|^p dt = 0. \tag{34}$$

Hence

$$(35)\quad \left[\int_0^{1-\delta} |f^{(k)}(t+\delta) - f^{(k)}(t)|^p dt\right]^{1/p} \leqq 2\|f^{(k)}(t) - \sigma_{2^n}(t)\|_p$$
$$+ \left[\int_0^{1-\delta} |\sigma_{2^n}(t+\delta) - \sigma_{2^n}(t)|^p dt\right]^{1/p} \leqq K'2^{1-n} + 2^{-n/p}\left[\sum_{i=1}^{2^n-1} |h_i^{(k)}|^p\right]^{1/p}$$

by (32)–(34). We now apply lemma 9 to (35) and find a constant C such that

$$\left[\int_0^{1-\delta} |f^{(k)}(t+\delta) - f^{(k)}(t)|^p dt\right]^{1/p} \leqq (2K' + C^{1/p})2^{-n}$$
$$\leqq (4K' + 2C^{1/p})\delta.$$

Hence

$$\left[\int_0^{1} |f^{(k)}(t+\delta) - f^{(k)}(t)| dt\right]^{1/p} = O(\delta)$$

and so $f^{(k)} \in \mathscr{L}_k^p$. This completes the proof of the theorem.

Remark. Other characterizations are possible, using the result [**2**] that $f \in \mathscr{L}_p^k$ if and only if $f^{(k)}$ is of *p-bounded variation* on [0, 1], i.e., the supremum over all subdivisions $\delta_n : 0 = x_0 < x_1 < \ldots < x_n = 1$ of the sum

$$\left[\sum_{i=0}^{n-1} |f^{(k)}(x_{i+1}) - f^{(k)}(x_i)|^p (x_{i+1} - x_i)^{1-p}\right]^{1/p}$$

if finite, and, for $p > 1$, the result [**5**] that f is of p-bounded variation on [0, 1] if and only if f is absolutely continuous and $f' \in L^p[0, 1]$.

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UNIQUE FACTORIZATION THEOREMS FOR SUBALGEBRAS OF THE INCIDENCE ALGEBRA

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1. Introduction. H. Scheid [4] has found necessary and sufficient conditions on a partially ordered set $S(\leqq)$ which is a direct sum of a countable number of trees for a certain subalgebra $G(+, *)$ of the incidence algebra $F(+, *)$ to be an integral domain. In this paper we prove that under similar conditions on S, $G(+, *)$ is actually a unique factorization domain or, failing this, that there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain and contains G. Similar results are then obtained as corollaries in the regular convolution rings of Narkiewicz.

2. Definitions and notations. Throughout this paper $S(\leqq)$ will denote a *locally finite partially ordered* set. By this we mean $\leqq$ is a partial ordering of the nonempty set S and for all $a, b \in S$ with $a \leqq b$ the *interval* $[a, b]$ of all $x \in S$ with $a \leqq x \leqq b$ is a finite set. A one-element interval is called a *unit interval*. A *chain* is a totally ordered set. If $C(\leqq)$ is a finite chain of n elements then the *length* $\lambda(C)$ of C is $n - 1$ while if C is infinite then $\lambda(C) = \infty$. A *tree* T is a partially ordered set with a least element such that every interval of T is a finite chain. A tree consisting of a single element is called *trivial*. The *length* $\lambda(T)$ of the tree T is the supremum of the lengths of all the intervals of T. The incidence algebra [5] on $S(\leqq)$ will be denoted by $F(+, *)$. The set F consists of all functions f from $S \times S$ to the field K such that $f(x, y) = 0$ if $x \nleqq y$. The addition in $F(+, *)$ is pointwise addition and the product is defined for $f, g \in F$ at (x, y) in $S \times S$ by

$$(f * g)(x, y) = \sum (f(x, z)g(z, y) : x \leqq z \leqq y).$$

The characteristic function of unit intervals $\epsilon \in F$ is the unity of $F(+, *)$ and $f \in F$ is a unit if and only if $x \in S$ implies $f(x, x) \neq 0$.

We shall use $\cong$ to denote isomorphism between both algebras and partially ordered sets. The field K will be of characteristic zero. We shall denote the ring of formal power series over K in a finite number n of indeterminants by K_n and in a countably infinite number of indeterminants by K_ω. The subalgebra $G(+, *)$ of $F(+, *)$ consists of all $g \in F$ such that if $x, y, u, v \in S$ with $[x, y] \cong [u, v]$ then $g(x, y) = g(u, v)$. Thus G consists of all functions in F

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which cannot distinguish between isomorphic intervals. If $I = [x, y]$ we shall write $f(x, y) = f(I)$. Note that $g \in G$ is a unit if and only if for some unit interval I_0, $g(I_0) \neq 0$.

3. The main theorems.

Scheid [4] proved the following.

THEOREM A. (a) *If* $S(\leqq) = T_1 \oplus \ldots \oplus T_n$ *is the direct sum of nontrivial trees* $T_1, \ldots, T_n$ *then* $G(+, *)$ *is an integral domain if and only if* $\lambda(T_i) = \infty$ *for* $1 \leqq i \leqq n$.

(b) *If* $S(\leqq) = T_1 \oplus T_2 \oplus \ldots$ *is the direct sum of a countably infinite number of trees* $T_1, T_2, \ldots$ *then* $G(+, *)$ *is an integral domain if and only if for each* T_i *with* $\lambda(T_i) < \infty$ *there are infinitely many* T_j *with* $\lambda(T_j) \geqq \lambda(T_i)$.

In proving our theorems we shall exhibit an isomorphism between the subalgebra and either K_n or K_ω. The ring K_n is known [7, p. 148] to be a unique factorization domain as is K_ω[1].

LEMMA. *If* $I = C_1 \oplus \ldots \oplus C_n$ *is the direct sum of finite nontrivial chains* $C_1, \ldots, C_n$ *then* I *satisfies the Jordan-Dedekind chain condition that all maximal chains* M *in* I *are of the same length, namely*

$$\lambda(M) = \lambda(I) = \lambda(C_1) + \ldots + \lambda(C_n).$$

If $I' = C_1' \oplus \ldots \oplus C_m'$ *then* $I \cong I'$ *if and only if* $m = n$ *and the number of* C_i' *of any given length equals the number of* C_i *of that length.*

Proof. The facts stated in this lemma are probably well known but for lack of a reference we prove the first part. Suppose $I = C_1 \oplus \ldots \oplus C_n$ as in the lemma and let $\lambda(C_i) = l_i$, $C_i = \{a_{i0}, a_{i1}, \ldots, a_{il_i}\}$, $1 \leqq i \leqq n$. Each a in I is then of the form

$$a = (a_{1j_1}, a_{2j_2}, \ldots, a_{nj_n})$$

where $0 \leqq j_i \leqq l_i$ for $1 \leqq i \leqq n$. The elements $a_0 = (a_{10}, a_{20}, \ldots, a_{n0})$ and $a_1 = (a_{1l_1}, a_{2l_2}, \ldots, a_{nl_n})$ are the least and greatest elements of I, respectively, and must be contained in any maximal chain of I. Let

$$a_0 = x_0 < x_1 < x_2 < \ldots < x_L = a_1$$

be a maximal chain of I. For convenience of notation let us think of an element $a \in I$ as an ordered n-tuple $(j_1, \ldots, j_n)$ of nonnegative integers. Thus $a_0 = (0, 0, \ldots, 0)$ and $a_1 = (l_1, \ldots, l_n)$. Now x_1 in the maximal chain must be an n-tuple of $n - 1$ zeros and one 1 for if not then there would be a y in I with $x_0 < y < x_1$ contrary to the maximality of the chain. Similarly x_2 must be an n-tuple of $n - 1$ zeros and a 2 or $n - 2$ zeros and two 1's. In general x_{k+1} is an n-tuple which agrees with x_k except that a 1 has been added to some entry of x_k where $0 \leqq k < L$. Thus $L = l_1 + \ldots + l_n = \lambda(C_1) + \ldots + \lambda(C_n)$ and all maximal chains in I are of the same length, $\lambda(I) = \lambda(C_1) + \ldots + \lambda(C_n)$.

The second part of the lemma follows from a known theorem [**6**, p. 200].

Our first theorem concerns the case when S is the direct sum of infinitely many trees but we impose a slightly stronger condition on S than Scheid did in part (b) of Theorem A.

THEOREM 1. *If* $S = T_1 \oplus T_2 \oplus \ldots$ *is the direct sum of a countably infinite number of trees* $T_1, T_2, \ldots$ *and if for each tree* T_i *and each interval* C *of* T_i *there are infinitely many* T_j *each containing an interval isomorphic to* C, *then* $G(+, *)$ *is a unique factorization domain.*

Proof. Since each interval of S is a direct sum of a finite number of chains of finite length it follows from the lemma that two intervals of S are isomorphic if and only if their representations as direct sums of chains contain the same number of chains of each possible positive length. Thus we may associate with each isomorphism equivalence class of intervals of S the sequence

$$r(I) = (r_1(I), r_2(I), \ldots) \tag{1}$$

where $r_j(I)$ is the number of chains of length j in the representation of I as a direct sum of chains. Thus two intervals I and I' of S are isomorphic if and only if $r(I) = r(I')$.

We divide the proof into two cases.

Case 1. Suppose there is a non-negative integer m such that $\lambda(T_i) \leqq m$ for all $i \geqq 1$ and $\lambda(T_i) = m$ for some $i \geqq 1$. Then the hypothesis of the theorem implies that $\lambda(T_i) = m$ for infinitely many $i \geqq 1$. If $m = 0$ then all the trees are trivial, S is a single element set and $F(+, *) = G(+, *) \cong K$, which is trivially a unique factorization domain. Thus assume that $m \geqq 1$.

Note that $r_j(I) = 0$ for all $j > m$ and all intervals I of S. We define π_j for $1 \leqq j \leqq m$ to be the characteristic function of chains of length j. Then we shall show that every $g \in G$ has a unique representation in the form

$$g = \sum a(i_1, \ldots, i_m)\pi_1^{i_1} * \ldots * \pi_m^{i_m} \tag{2}$$

where the summation extends over all m-tuples $(i_1, \ldots, i_m)$ of nonnegative integers, $a(i_1, \ldots, i_m) \in K$, $\pi_j^{i_j} = \pi_j * \ldots * \pi_j$ with i_j factors if $i_j \geqq 1$ and $\pi_j^0 = \epsilon$. Once this representation has been established, there is an immediate isomorphism between $G(+, *)$ and the formal power series ring K_m. For if $x_1, \ldots, x_m$ are the indeterminants in K_m, simply correspond π_j to x_j for $1 \leqq j \leqq m$.

If $(i_1, \ldots, i_m)$ is a fixed m-tuple of nonnegative integers and I is an interval of S then we claim that

$$\pi_1^{i_1} * \ldots * \pi_m^{i_m}(I) = 0$$

unless

$$\lambda(I) = i_1 + 2i_2 + \ldots + mi_m. \tag{3}$$

Recall from the proof of the lemma that

$$\lambda(I) = r_1(I) + 2r_2(I) + \ldots + mr_m(I).$$

To prove the claim suppose $I = [a, b]$ and that

$$\pi_1^{i_1} * \ldots * \pi_m^{i_m}(I) \neq 0.$$

Then there is a chain, $a = x_0 \leqq x_1 \leqq \ldots \leqq x_m = b$ in I such that

$$\pi_j^{i_j}\,(x_{j-1}, x_j) \neq 0$$

for $1 \leqq j \leqq m$. Thus $\lambda(x_{j-1}, x_j) = ji_j$ and the chain $x_0 \leqq x_1 \leqq \ldots \leqq x_m$ refines to a maximal chain in I of length $i_1 + 2i_2 + \ldots + mi_m$ which equals $\lambda(I)$ by the lemma. This proves the claim.

Thus if the series in (2) is evaluated at an interval I the only terms which can be nonzero are those for which the m-tuple $(i_1, \ldots, i_m)$ satisfies (3) and these are finite in number. In fact the number of solutions $(i_1, \ldots, i_m)$ of (3) is the number of partitions of the integer $\lambda(I)$ into positive integers less than or equal to m.

Now let L be a fixed nonnegative integer and consider all intervals I of S such that $\lambda(I) = L$. Since we are not distinguishing between isomorphic intervals we need only consider those m-tuples $(r_1(I), \ldots, r_m(I))$ for which $r_1(I) + 2r_2(I) + \ldots + mr_m(I) = L$. We now arrange all such m-tuples as follows:

$$\begin{aligned}
&(L, 0, 0, 0, \ldots, 0),\ (L-2, 1, 0, 0, \ldots, 0), \ldots, (0 \text{ or } 1, [L/2], 0, 0, \ldots, 0),\\
&(L-3, 0, 1, 0, \ldots, 0),\ (L-5, 1, 1, 0, \ldots, 0), \ldots, (0 \text{ or } 1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad [(L-3)/2], 1, 0, 0, \ldots, 0)\\
&(L-6, 0, 2, 0, \ldots, 0),\ (L-8, 1, 2, 0, \ldots, 0), \ldots, (0 \text{ or } 1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad [(L-6)/2], 2, 0, \ldots, 0),\\
&\qquad\qquad\qquad \ldots\\
&(0 \text{ or } 1 \text{ or } 2, 0, [L/3], 0, \ldots, 0),\ (0 \text{ or } 1, 0 \text{ or } 1, [L/3], 0, \ldots, 0),\\
&(L-4, 0, 0, 1, 0, \ldots, 0),\ (L-6, 1, 0, 1, 0, \ldots, 0), \ldots, (0 \text{ or } 1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad [(L-4)/2], 0, 1, 0, \ldots, 0)\\
&(L-7, 0, 1, 1, 0, \ldots, 0),\ (L-9, 1, 1, 1, 0, \ldots, 0), \ldots, (0 \text{ or } 1,\\
&\qquad\qquad\qquad\qquad\qquad\qquad [(L-7)/2], 1, 1, 0, \ldots, 0)\\
&\qquad\qquad\qquad \ldots
\end{aligned}$$

where the brackets denote the greatest integer function.

As an example, if $m = 4$ and $L = 7$ the above arrangement is

$$\begin{aligned}
&(7, 0, 0, 0),\ (5, 1, 0, 0),\ (3, 2, 0, 0),\ (1, 3, 0, 0),\\
&(4, 0, 1, 0),\ (2, 1, 1, 0),\ (0, 2, 1, 0),\\
&(1, 0, 2, 0),\\
&(3, 0, 0, 1),\ (1, 1, 0, 1),\\
&(0, 0, 1, 1).
\end{aligned}$$

For any given m-tuple $(i_1, \ldots, i_m)$ the hypothesis of the theorem ensures the existence of an interval I of S with $r(I) = (i_1, i_2, \ldots, i_m, 0, 0, \ldots)$. We now select intervals $I_1, I_2, \ldots$ such that $\lambda(I_j) = L$ and the m-tuples $(r_1(I_j), r_2(I_j), \ldots, r_m(I_j))$ for $j = 1, 2, \ldots$ are the m-tuples in the above arrangement, i.e.,

$$\begin{aligned}(r_1(I_1), r_2(I_1), \ldots, r_m(I_1)) &= (L, 0, 0, \ldots, 0)\\ (r_1(I_2), r_2(I_2), \ldots, r_m(I_2)) &= (L-2, 1, 0, \ldots, 0)\\ &\cdots\end{aligned}$$

We now proceed to evaluate (2) successively at $I_1, I_2, \ldots$ forcing equality and thereby determining the coefficients $a(i_1, \ldots, i_m)$:

$$g(I_1) = \sum a(i_1, \ldots, i_m)\, \pi_1^{i_1} * \ldots * \pi_m^{i_m}\, (I_1)$$

and since the only subintervals of I_1 which are chains are chains of length 1 we have

$$g(I_1) = a(L, 0, 0, \ldots, 0)\, \pi_1^{L}\, (I_1),$$

and

$$a(L, 0, 0, \ldots, 0) = g(I_1)/(\pi_1^{L}\, (I_1)).$$

Actually $\pi_1^{L}(I_1) = L!$ but we only need that it is nonzero. Continuing,

$$g(I_2) = \sum a(i_1, \ldots, i_m)\, \pi_1^{i_1} * \ldots * \pi_m^{i_m}(I_2)$$

and since the only subintervals of I_2 which are chains are chains of length 1 or 2, we have

$$g(I_2) = \sum a(i_1, i_2, 0, 0, \ldots, 0)\, \pi_1^{i_1} * \pi_2^{i_2}(I_2).$$

There is only one subinterval of I_2 which is a chain of length 2 and thus

$$g(I_2) = a(L, 0, 0, \ldots, 0)\pi_1^{L}(I_2) + a(L-2, 1, 0, \ldots, 0)\pi_1^{L-2} * \pi_2(I_2).$$

Since $\pi_1^{L-2} * \pi_2(I_2) \neq 0$, the coefficient $a(L-2, 1, 0, \ldots, 0)$ is thus determined. Continuing in this way we can evaluate all the coefficients $a(i_1, \ldots, i_m)$ for which

$$i_1 + 2i_2 + \ldots + mi_m = L. \tag{4}$$

Since every m-tuple $(i_1, \ldots, i_m)$ satisfies (4) for one and only one L this procedure uniquely determines all the coefficients in (2) and completes the proof of this case.

Case 2. Suppose that for each integer $m \geqq 1$ there is a tree T_t which contains an interval of length m. In this case we claim that each $g \in G$ has a unique representation in the form

$$g = \sum a(i_1, i_2, \ldots)\, \pi_1^{i_1} * \pi_2^{i_2} * \ldots \tag{5}$$

where the summation now extends over all sequences $(i_1, i_2, \ldots)$ of nonnegative integers all but a finite number of which are zero.

Once this representation is established the isomorphism between $G(+, *)$ and K_ω is immediate.

We throw the proof back to case 1 by considering for each nonnegative integer L all sequences $(i_1, i_2, \ldots)$ satisfying

$$i_1 + 2i_2 + \ldots = L. \tag{6}$$

The number of solutions $(i_1, i_2, \ldots)$ of (6) is the number of unrestricted partitions of L into positive integers and in any solution $i_j = 0$ for $j > L$. Thus if we take $m = L$ in case 1 the solutions of (6) coincide with the solutions of (4) and the coefficients $a(i_1, \ldots, i_L, 0, 0, \ldots)$ in (5) can be determined as in case 1. When the series in (5) is evaluated at an interval of S it reduces to a finite sum as in case 1 and equals the value of g at that interval.

In part (a) of Theorem A, Scheid found necessary and sufficient conditions for $G(+, *)$ to be an integral domain if S is the direct sum of a finite number of trees. We have not been able to prove (or disprove) that such a G is a unique factorization domain but we do have the following.

THEOREM 2. *If $S = T_1 \oplus \ldots \oplus T_n$ is the direct sum trees with $\lambda(T_i) = \infty$ for $1 \leqq i \leqq n$ then there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain and contains $G(+, *)$.*

Proof. Each interval I of S is of the form

$$I = C_1 \oplus \ldots \oplus C_n \tag{7}$$

where C_i is a chain from T_i for $1 \leqq i \leqq n$. Define

$$\Lambda(I) = (\lambda(C_1), \ldots, \lambda(C_n)) \tag{8}$$

where C_i is given by (7). Note that $\Lambda(I) = \Lambda(I')$ implies $I \cong I'$ but not conversely. We now define for each n-tuple $(i_1, \ldots, i_n)$ of nonnegative integers the function $e(i_1, \ldots, i_n)$ to be the characteristic function of those intervals I of S with $\Lambda(I) = (i_1, \ldots, i_n)$. Then $e(i_1, \ldots, i_n) \in G$ if and only if $i_1 = i_2 = \ldots = i_n$, for if $\Lambda(I) = (i_1, \ldots, i_n)$ and $i_j \neq i_k$ for some j, k with $1 \leqq j < k \leqq n$, say for convenience, $i_1 \neq i_2$ then by choosing I' with $\Lambda(I') = (i_2, i_1, i_3, \ldots, i_n)$ we have $I \cong I'$ but $e(i_1, \ldots, i_n)\ (I) = 1$ while $e(i_1, \ldots, i_n)\ (I') = 0$.

We now define the subset H of F to consist of all $h \in F$ representable in the form

$$h = \sum a(i_1, \ldots, i_n) e(i_1, \ldots, i_n) \tag{9}$$

where the summation extends over all n-tuples $(i_1, \ldots, i_n)$ of nonnegative integers and $a(i_1, \ldots, i_n) \in K$.

We claim that any pair of functions $e(i_1, \ldots, i_n)$, $e(k_1, \ldots, k_n)$ multiply according to the rule

$$e(i_1, \ldots, i_n) * e(k_1, \ldots, k_n) = e(i_1 + k_1, \ldots, i_n + k_n). \tag{10}$$

For suppose for some interval $I = [a, b]$ of S that

$$e(i_1, \ldots, i_n) * e(k_1, \ldots, k_n)(I) \neq 0. \tag{11}$$

If $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ then there is an $x = (x_1, \ldots, x_n) \in S$ with a $\leqq x \leqq b$ such that

$$e(i_1, \ldots, i_n)(a, x) \neq 0 \text{ and } e(k_1, \ldots, k_n)(x, b) \neq 0.$$

Thus

$$\Lambda(a, x) = (\lambda(a_1, x_1), \ldots, \lambda(a_n, x_n)) = (i_1, \ldots, i_n)$$

and

$$\Lambda(x, b) = (\lambda(x_1, b_1), \ldots, \lambda(x_n, b_n)) = (k_1, \ldots, k_n)$$

and hence

$$\Lambda(I) = (i_1 + k_1, \ldots, i_n + k_n).$$

Furthermore the above x is unique and there is exactly one nonzero term in the convolution (11). Thus (10) is established.

Because of (10) there is an isomorphism between $H(+, *)$ and the power series ring K_n if we make the correspondence

$$\sum a(i_1, \ldots, i_n)e(i_1, \ldots, i_n) \leftrightarrow \sum a(i_1, \ldots, i_n)x_1{}^{i_1} \ldots x_n{}^{i_n}.$$

To complete the proof we must show that $G \subset H$. To this end we define for each n-tuple $(i_1, \ldots, i_n)$ the function

$$\bar{e}(i_1, \ldots, i_n) = \sum e(\pi(i_1), \ldots, \pi(i_n)) \tag{12}$$

where the summation extends over all distinct permutations $(\pi(i_1), \ldots, \pi(i_n))$ of $(i_1, \ldots, i_n)$. For example

$$\bar{e}(i, i, \ldots, i) = e(i, i, \ldots, i)$$

and

$$\bar{e}(1, 0, 0, \ldots, 0) = e(1, 0, 0, \ldots, 0) + e(0, 1, 0, \ldots, 0) + \ldots + e(0, 0, 0, \ldots, 0, 1).$$

Then each $\bar{e} \in G$ and if $g \in G$ then g is expressible in the form

$$g = \sum b(i_1, \ldots, i_n)\bar{e}(i_1, \ldots, i_n). \tag{13}$$

To see this, note that $\bar{e}(i_1, \ldots, i_n)$ is the characteristic function of intervals isomorphic to the interval I with $\Lambda(I) = (i_1, \ldots, i_n)$ and the coefficient $b(i_1, \ldots, i_n)$ in (13) is simply $g(I)$. Since each $\bar{e}$ is a finite sum of the e's, (13) implies that $G \subset H$ and the proof is complete.

Equation (13) leads one to suspect that $G(+, *)$ is isomorphic to K_n also but the $\bar{e}$ functions do not have a "nice multiplication rule". Perhaps some other choice of "generating functions" in place of the $\bar{e}$'s would produce a representation of the type in (13) and have a "nice multiplication rule" but we have been unable to find such functions.

We return now to the case when S is the direct sum of infinitely many trees. In Theorem 1 we imposed a stronger condition on S than Scheid did in part (*b*) of Theorem A. Those S which satisfy Scheid's condition but not ours all satisfy

Condition B. Suppose $S = T_1 \oplus T_2 \oplus \ldots$ is the direct sum of trees $T_1, T_2, \ldots$, a finite number of which, say $T_1, \ldots, T_n$, are of infinite length while there is an integer $m \geqq 0$ such that $\lambda(T_i) \leqq m$ for all $i > n$ and $\lambda(T_i) = m$ for infinitely many $i > n$.

THEOREM 3. *If S satisfies Condition B then there is a subalgebra $H(+, *)$ of $F(+, *)$ which is a unique factorization domain isomorphic to K_{m+n} and contains G.*

Proof. The method of proof is a combination of the methods used in proving Theorems 1 and 2.

Each interval I of S is of the form

$$I = C_1 \oplus C_2 \oplus \ldots \oplus C_n \oplus C_{n+1} \oplus \ldots \tag{14}$$

where C_i is a chain from T_i for $i \geqq 1$ and all but a finite number of the C_i are trivial one element chains. Since $T_1, \ldots, T_n$ are of infinite length the first n chains in (14) may be of any arbitrary length while $\lambda(C_i) \leqq m$ for all $i > n$. For each interval I of S we define

$$\Lambda'(I) = (r_1'(I), \ldots, r_m'(I), \lambda(C_1), \ldots, \lambda(C_n)) \tag{15}$$

where $r_j'(I)$ is the number of chains C_i in (14) of length j with $i > n$. We define π_j' for $1 \leqq j \leqq m$ to be the characteristic function of intervals I of S with

$$\Lambda'(I) = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where the 1 appears in position j. Thus π_j' is the characteristic function of a chain of length j which is an interval in a tree T_i with $i > n$.

We also define for each n-tuple $(k_1, \ldots, k_n)$ of nonnegative integers the function $e'(k_1, \ldots, k_n)$ to be the characteristic function of intervals I of S with

$$\Lambda'(I) = (0, 0, \ldots, 0, k_1, k_2, \ldots, k_n).$$

Note that none of the π_j''s or the e''s are in G except for $e'(0, 0, \ldots, 0) = \epsilon$.

We then let H be the set of all $h \in F$ representable in the form

$$h = \sum a(i_1, \ldots, i_{m+n})\pi_1'^{i_1} * \ldots \pi_m'^{i_m} e'(i_{m+1}, \ldots, i_{m+n}), \tag{16}$$

where the summation extends over all $m + n$-tuples $(i_1, \ldots, i_{m+n})$ of nonnegative integers and the coefficients $a(i_1, \ldots, i_{m+n}) \in K$.

As in the proof of Theorem 2, two e' functions multiply according to

$$e'(k_1, \ldots, k_n) * e'(k_1', \ldots, k_n') = e'(k_1 + k_1', \ldots, k_n + k_n')$$

and $H(+, *)$ is isomorphic to K_{m+n}.

The proof will be complete if we show that $G \subset H$. To see this observe for $1 \leqq j \leqq m$,

$$\pi_j = \pi_j' + e'(j, 0, 0, \ldots, 0) + e'(0, j, 0, \ldots, 0) + \ldots + e'(0, 0, 0, \ldots, 0, j) \tag{17}$$

and for $j > m$,

$$\pi_j = e'(j, 0, 0, \ldots, 0) + e'(0, j, 0, \ldots, 0) + \ldots + e'(0, 0, \ldots, 0, j). \tag{18}$$

We then claim that each $g \in G$ has a representation of the form

$$g = \sum b(i_1, i_2, \ldots)\pi_1^{i_1} * \pi_2 *^{i_2} \ldots \tag{19}$$

where $b(i_1, i_2, \ldots) \in K$ and the summation extends over all sequences $(i_1, i_2, \ldots)$ of nonnegative integers all but a finite number of which are zero. Once (19) has been established we are done since (17), (18) and (19) imply $G \subset H$.

The coefficients $b(i_1, \ldots, i_m, 0, 0, \ldots)$ in (19) can be evaluated by the method used in case 1 of the proof of Theorem 1. The difficulty in extending the same method to evaluate all the $b(i_1, i_2, \ldots)$ is that not all possible direct sums of chains appear in S. For example, no interval of S is the direct sum of $n + 1$ chains each of length $m + 1$. However, this is not a serious difficulty for our present purpose. It merely allows some freedom in selecting the coefficients $b(i_1, i_2, \ldots)$ for which no interval I exists in S with $r(I) = (i_1, i_2, \ldots)$. In fact all such b's may be chosen arbitrarily. Thus each $g \in G$ has a representation of the type (19) although this representation is not unique. This completes the proof of Theorem 3.

If $S = T_1 \oplus T_2 \oplus \ldots$ is the direct sum of trees and $\lambda(T_i) = 1$ for all $i \geqq 1$ then $G(+, *) \cong K_1$ and every element $g \in G$ is uniquely expressible in the form $\sum(a_n \pi_1^n : n \geqq 0)$. Furthermore π_1 is the only prime element (up to associates) in G and every $g \in G$ is of the form $u * \pi_1^{n_0}$ where u is a unit of G and n_0 is the minimum n such than $a_n \neq 0$. In contrast to this situation we have the following.

THEOREM 4. *If $S(\leqq)$ is a locally finite partially ordered set which contains intervals C_1 and C_2 which are chains of length* 1 *and* 2 *respectively, and if S also contains an interval isomorphic to $C_1 \oplus C_1$, then $G(+, *)$ contains infinitely many irreducible elements, no two of which are associates.*

Proof. We claim that the set

$$P = \{\pi_1 + a\pi_2 : a \in K\}$$

is a set of nonassociated irreducible elements. Suppose that

$$\pi_1 + a\pi_2 = g_1 * g_2$$

where g_1 and g_2 are nonunits of G. Then (recall that nonunits of G vanish on unit intervals of S)

$$1 = (\pi_1 + a\pi_2)(C_1) = (g_1 * g_2)(C_1) = 0,$$

a contradiction. Thus $\pi_1 + a\pi_2$ is irreducible. Now suppose that two elements $\pi_1 + a\pi_2$ and $\pi_1 + b\pi_2$ of P are associates, say

$$\pi_1 + a\pi_2 = g * (\pi_1 + b\pi_2)$$

where g is a unit of G. Then using I_0 to denote a unit interval and evaluating this equation successively at C_1, C_2 and $C_1 \oplus C_1$ we obtain

$$1 = g(I_0),\ a = bg(I_0) + g(C_1) \text{ and } 0 = 2g(C_1).$$

Thus $a = b$ and the proof is complete.

4. Applications to regular convolution rings. Let $A(+, *_C)$ be a regular convolution ring as defined by Narkiewicz [**2**], i.e., A is the set of arithmetic functions, $+$ is pointwise addition and $*_C$ is a convolution product defined as follows. Let C be a mapping from the natural numbers N to the finite subsets of N such that $C(n)$ is a subset of the set of divisors of n for each $n \in N$. Then for $\alpha, \beta \in A$ and $n \in N$,

$$(\alpha *_C \beta)(n) = \sum (\alpha(d)\beta(n/d) : d \in C(n)).$$

Then Narkiewicz calls $A(+, *_C)$ regular if it is a commutative ring with unity, the multiplicative functions form a semigroup under $*_C$ and the "Möbius" function assumes only the values 0 or -1 at a prime power $p^k > 1$. As Narkiewicz showed [**2**, Theorem II], every regular convolution is determined by selecting for each prime $p \in N$ a collection of arithmetic progressions of the type $\{m, 2m, 3m, \ldots\}$ (finite or infinite) which partition N and then defining for each power p^k, $C(p^k) = \{1, p^m, p^{2m}, \ldots, p^{rm}\}$ where $k = rm$ and $\{m, 2m, \ldots\}$ is the progression in which k appears.

Let $\leqq_C$ be the relation on N defined by: $a \leqq_C b$ if and only if $a \in C(b)$. Then $N(\leqq_C)$ is a locally finite partially ordered set and can be expressed as a direct sum of trees as Scheid has shown [**3**]. For each prime $p \in N$ let $T_p = \{1, p, p^2, \ldots\}$. Then $T_p(\leqq_C)$ is a tree and

$$N(\leqq_C) = T_2 \oplus T_3 \oplus T_5 \oplus \ldots \oplus T_{p_n} \oplus \ldots \tag{20}$$

where p_n is the nth prime. Let $F(+, *)$ be the incidence algebra on $N(\leqq_C)$ and $G(+, *)$ be the subalgebra defined previously. Then $A(+, *_C)$ can be imbedded in $F(+, *)$ under the mapping $\alpha \leftrightarrow \alpha'$ where α' satisfies $\alpha'(m, n) = \alpha'(1, n/m) = \alpha(n/m)$ for all $m, n \in N$ such that $m \in C(n)$. We shall denote the image of $A(+, *_C)$ in $F(+, *)$ by $A(+, *)$. Then it is easily seen that $G(+, *) \subset A(+, *)$ and the following theorems are then corollaries of the theorems in section 3. We shall use the notation $\lambda(p^k) = \lambda(1, p^k)$, i.e., the length of the prime power p^k is the length of the chain $[1, p^k]$ in the tree T_p.

THEOREM 5. *If* $N(\leqq_C)$ *is given by* (20) *and for each prime power* p^k, *there are infinitely many primes* q *such that* $\lambda(q^l) \geqq \lambda(p^k)$ *for some power* q^l, *then* $G(+, *)$ *is a unique factorization domain.*

THEOREM 6. *If* $N(\leqq_C)$ *is given by* (20) *and there are positive integers* m *and* n *such that* $\lambda(T_{p_i}) = \infty$ *for* n *values of* i, $\lambda(T_{p_i}) \leqq m$ *for all other* i *and* $\lambda(T_{p_i}) = m$ *for infinitely many* i, *then* $A(+, *)$ *contains a subalgebra* $H(+, *)$ *which is a unique factorization domain isomorphic to* K_{m+n} *and contains* $G(+, *)$.

THEOREM 7. *If* $N(\leqq_C)$ *is given by* (20) *and if* $\lambda(T_{p_i}) \geqq 2$ *for some* i *then* $G(+, *)$ *contains infinitely many nonassociated irreducible elements.*

The only thing requiring a proof in the above theorems is that $H \subset A$ in Theorem 6. Going back to the definition of H in Section 3, this amounts to showing that the π_j''s and the e''s lie in A and it is not difficult to verify this.

An interesting special case is when each Tp_n is an infinite chain and $\leqq_C$ is the usual divisibility order on N. Then $A(+, *)$ is the Dirichlet convolution ring which Cashwell and Everett have shown to be isomorphic to K_ω. Thus by Theorem 5, $A(+, *)$ is isomorphic to its proper subring $G(+, *)$.

Another case of interest is when $\lambda(Tp_n) = 1$ for all $n \geqq 1$. Then $A(+, *)$ is the unitary convolution ring of arithmetic functions and $G(+, *) \cong K_1$.

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ON OPEN PROJECTIONS OF GCR ALGEBRAS

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Throughout this paper $\mathfrak{A}$ will denote a C^*-algebra and $\mathfrak{B}$ will denote its second dual, which is identified with the enveloping von Neumann algebra of $\mathfrak{A}$. A projection E in $\mathfrak{B}$ is said to be *open* if it supports a left ideal in $\mathfrak{A}$, that is, if $\mathfrak{B}E = \mathfrak{J}^-$ for some left ideal $\mathfrak{J}$ in $\mathfrak{A}$. Here the bar $^-$ means the *stong* closure. When $\mathfrak{A}$ has a unit, this definition coincides with the definition of Akemann [**1**, Definition II.1]. In the sequel, we shall solely be concerned with two-sided ideals, and consequently central projections [**4**, I, § 3, Corollary 3 of Theorem 2]. Our aim is to show that $\mathfrak{A}$ is CCR if and only if the open central projections are strongly dense in the set of central projections on $\mathfrak{B}$. If $\mathfrak{A}$ is a GCR algebra, this implies that the complete Boolean lattice generated by the open central projections of $\mathfrak{B}$ is the set of central projections of $\mathfrak{B}$. As a corollary we get the result that every ideal preserving automorphism of a GCR algebra is weakly inner. This was proved by Lance [**4**] for separable GCR algebras and recently extended to arbitrary GCR algebras by Elliott [**5**] using methods different from ours.

Before we begin we fix some notation. Let $\mathfrak{P}$ be the complete Boolean algebra of central projections in $\mathfrak{B}$, let $\mathfrak{P}_0$ be the set of all open central projections in $\mathfrak{B}$, and let $\langle\mathfrak{P}_0\rangle$ be the complete Boolean algebra generated by $\mathfrak{P}_0$ in $\mathfrak{P}$. For every $A \in \mathfrak{B}$ let C_A denote the orthogonal complement of the largest $P \in \mathfrak{P}$ such that $AP = 0$. If $A \in \mathfrak{A}$, then C_A is the support of the two-sided ideal in $\mathfrak{A}$ that is generated by A and thus the projection C_A is in $\mathfrak{P}_0$. Let $\mathfrak{J}_a$ be the closed two-sided ideal of $\mathfrak{B}$ generated by the abelian projections of $\mathfrak{B}$.

LEMMA 1. *Let π be a non-degenerate representation of the CCR algebra $\mathfrak{A}$ on a Hilbert space H, let $A \in \mathfrak{A}$, and let P be a central projection of the von Neumann algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$. Then there is a net $\{Q_n\}$ of central projections of $\pi(\mathfrak{A})''$ which converge strongly to* 1 *and a corresponding net $\{A_n\}$ in $\pi(\mathfrak{A})$ such that $\pi(A)PQ_n = A_nQ_n$ for every Q_n.*

Proof. There is a normal homomorphism ϕ of the enveloping von Neumann algebra $\mathfrak{B}$ of $\mathfrak{A}$ onto $\pi(\mathfrak{A})''$ such that $\phi(B) = \pi(B)$ for every $B \in \mathfrak{A}$ [**3**, 12.1.5]. Because ϕ maps the centre of $\mathfrak{B}$ onto the centre of $\pi(\mathfrak{A})''$ and since ϕ is strongly continuous on bounded subsets of $\mathfrak{B}$, there is no loss generality in the assumption that π is the canonical embedding of $\mathfrak{A}$ in $\mathfrak{B}$ and that H is the canonical Hilbert space of $\mathfrak{B}$.

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Now it is sufficient to find, for every finite subset $\mathfrak{X}$ of H and every $\epsilon > 0$, a projection $Q \in \mathfrak{P}$ and an element $A' \in \mathfrak{A}$ such that $APQ = A'Q$ and such that $||x - Qx|| \leqq \epsilon$ for every $x \in \mathfrak{X}$. Let $\{P_i | 0 \leqq i \leqq i_0\}$ be a subset of $\mathfrak{P}_0$ indexed by the ordinals such that

(i) $P_0 = 0$, $P_{i_0} = 1$;
(ii) $P_i < P_{i+1}$ $(i < i_0)$;
(iii) $\operatorname{lub}\{P_i | i < j\} = P_j$ if j is a limit ordinal; and
(iv) $\mathfrak{B}(P_{i+1} - P_i)$ is the strong closure of a closed two-sided ideal in $\mathfrak{A}(1 - P_i)$ generated by elements in $\mathfrak{A}(1 - P_i)$ that are contained in $\mathfrak{J}_a$ [**6**, pp. 148–149].

Setting $P_i' = P_{i+1} - P_i$ for all $i < i_0$, we obtain a set of orthogonal central projections of sum 1. There are ordinals $\{i(j) | 1 \leqq j \leqq n\}$ with $i(n) < i(n - 1) < \ldots < i(1) < i_0$ such that

$$||\textstyle\sum_j P_{i(j)}'x - x|| \leqq \epsilon/3$$

for every $x \in \mathfrak{X}$. There is also an orthogonal set $\{R_k\}$ of central projections of sum 1 and a corresponding bounded set $\{B_k\}$ of elements of $\mathfrak{J}_a$ such that $A = \sum B_k R_k$ [**6**, Corollary, Theorem 6]. Consequently, there is a finite subset of the R_k of sum R such that

$$||Rx - x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. We have that $AR \in \mathfrak{J}_a$ and so $APR \in \mathfrak{J}_a$. We may find an element $A_1 \in \mathfrak{A}$ such that $A_1(1 - P_{i(1)+1}) = 0$ and a projection Q_1' in $\mathfrak{P}$ majorized by $RP_{i(1)}'$ with $||(Q_1' - P_{i(1)}')x|| \leqq \epsilon/3n$ for every $x \in \mathfrak{X}$ such that $APRQ_1' = A_1Q_1'$ [**6**, Lemma 4]. Setting $Q_1 = RQ_1'$, we obtain a central projection Q_1 majorized by $P_{i(1)}'$ with

$$||(P_{i(1)}' - Q_1)x|| \leqq ||(P_{i(1)}' - Q_1')x|| + ||Q_1'||\,||(1 - R)x|| \leqq 2\epsilon/3n$$

for every $x \in \mathfrak{X}$ such that $APQ_1 = A_1Q_1$. Continuing by recursion, we obtain central projections Q_j majorized by $P_{i(j)}'$ with

$$||(P_{i(j)}' - Q_j)x|| \leqq 2\epsilon/3n$$

for every $x \in \mathfrak{X}$ and elements A_j in $\mathfrak{A}$ with $A_j(1 - P_{i(j)+1}) = 0$ such that

$$(AP - (A_1 + \ldots + A_{j-1}))Q_j = A_jQ_j$$

for $2 \leqq j \leqq n$. Setting $Q = \sum Q_j$ and $A' = \sum A_j$, we obtain a central projection Q with

$$||x - Qx|| \leqq ||(1 - \textstyle\sum_j P_{i(j)}')x|| + \sum ||(P_{i(j)}' - Q_j)x|| \leqq \epsilon$$

for every $x \in \mathfrak{X}$ such that $APQ = A'Q$.

We can now obtain the following theorem.

THEOREM 2. *A C^*-algebra $\mathfrak{A}$ is a CCR algebra if and only if $\mathfrak{P}_0$ is strongly dense in $\mathfrak{P}$.*

Proof. Let $\mathfrak{A}$ be a CCR algebra. Since $\mathfrak{P}_0^-$ is clearly closed under taking least upper bounds of monotonely increasing nets, it is sufficient to show that every nonzero $P \in \mathfrak{P}$ majorizes a nonzero Q in $\mathfrak{P}_0^-$. We proceed to do this. Since $\mathfrak{A}$ is strongly dense in $\mathfrak{B}$, there is an $A \in \mathfrak{A}$ such that $AP \neq 0$. There is a net $\{Q_n\}$ of projections in $\mathfrak{P}$ which converges strongly to 1 and a corresponding net $\{A_n\}$ of elements of $\mathfrak{A}$ such that $APQ_n = A_nQ_n$ (Lemma 1). But the nonzero projection C_AP is in $\mathfrak{P}_0^-$ since $C_APQ_n = C_{A_n}Q_n$ and thus

$$\lim\|(C_AP - C_{A_n})x\| \leqq \lim \sup\|C_AP - C_{A_n}\| \, \|(1 - Q_n)x\| = 0$$

for every x in the Hilbert space of $\mathfrak{B}$. This completes the proof of this part of the theorem.

Conversely, let $\mathfrak{P}_0^- = \mathfrak{P}$. Let $\mathfrak{B}$ be considered as a module over its centre $\mathfrak{Z}$. Let ϕ be a bounded module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$ and let ψ be the unique extension of the restriction of ϕ to $\mathfrak{A}$ to a σ-weakly continuous linear function of $\mathfrak{B}$ into $\mathfrak{Z}$ [**6**, Proposition 1]. It is sufficient to show that ψ is a module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$ [**6**, Theorem 6]. If $P \in \mathfrak{P}_0$, then there is a monotonely increasing net $\{A_n\}$ in $\mathfrak{A}^+$ which converges σ-strongly to P [**1**, III.4]. Then for every $A \in \mathfrak{A}$ we have that

$$(1 - P)\psi(PA) = \lim(1 - P)\psi(A_nA) = \lim(1 - P)\phi(A_nA) = \lim \phi((1 - P)A_nA) = 0$$

and hence $\psi(PA) = P\psi(PA)$. Here the limits are taken in the σ-weak topology. Since $\mathfrak{A}$ is σ-weakly dense in $\mathfrak{B}$, $\mathfrak{P}_0$ is a bounded and σ-strongly dense in $\mathfrak{P}$, and ψ is σ-weakly continuous, we see that $\psi(PA) = P\psi(PA)$ for every $A \in \mathfrak{B}$ and $P \in \mathfrak{P}$. But this means that

$$\psi(PA) = P\psi(PA) + P(1 - P)\psi((1 - P)A) = P\psi(PA) + P\psi((1 - P)A) = P\psi(A)$$

for every $A \in \mathfrak{B}$ and $P \in \mathfrak{P}$. Now because ψ is a uniformly continuous linear function into $\mathfrak{Z}$ and because elements of $\mathfrak{Z}$ are uniform limits of linear combinations of projections of $\mathfrak{Z}$, we obtain that $\psi(CA) = C\psi(A)$ for every $A \in \mathfrak{B}$ and $C \in \mathfrak{Z}$. Hence ψ is a module homomorphism of $\mathfrak{B}$ into $\mathfrak{Z}$.

We now consider GCR algebras.

THEOREM 3. *If $\mathfrak{A}$ is a GCR algebra, then $\langle\mathfrak{P}_0\rangle = \mathfrak{P}$.*

Proof. It is sufficient to show that $\langle\mathfrak{P}_0\rangle^- = \mathfrak{P}$ since the complete Boolean algebra of projections $\langle\mathfrak{P}_0\rangle$ contains every projection in the weakly closed algebra which it generates [**2**, Theorem 2.8]. Given $P \in \mathfrak{P}$, a finite subset $\mathfrak{X}$ of the Hilbert space H of $\mathfrak{B}$, and $\epsilon > 0$, it is sufficient to find a Q in $\langle\mathfrak{P}_0\rangle^-$ such that $\|(P - Q)x\| \leqq \epsilon$ for every $x \in \mathfrak{X}$. Let $\{P_i | 0 \leqq i \leqq i_0\}$ be a maximal set of central projections indexed by the ordinals such that

(i) $P_0 = 0$;

(ii) $P_i < P_{i+1}$ $(i < i_0)$;

(iii) $\operatorname{lub}\{P_i | i < j\} = P_j$ if j is a limit ordinal; and

(iv) $\mathfrak{B}(P_{i+1} - P_i)$ is the strong closure of a CCR ideal $\mathfrak{J}_i$ contained in $\mathfrak{A}(1 - P_i)$ (cf. proof of Lemma 1).

Then clearly we must have $P_{i_0} = 1$.

We notice that each P_i is in $\mathfrak{P}_0$ since P_i supports the ideal

$$\{A \in \mathfrak{A} | A(1 - P_i) = 0\}.$$

Thus, every projection $P_i' = P_{i+1} - P_i$ is in $\langle \mathfrak{P}_0 \rangle$. Since $\sum P_i' = 1$, we may find ordinals $i(1), \ldots, i(n)$ such that $||(1 - \sum_j P_{i(j)}')x|| \leqq \epsilon/3$ for every $x \in \mathfrak{X}$. Now $\mathfrak{J}_{i(j)}$ is strongly dense in $\mathfrak{B}P_{i(j)}'$, and thus there is positive element A_j in the unit sphere of $\mathfrak{J}_{i(j)}$ with

$$||(A_j - P_{i(j)})x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. Therefore, we have

$$||(C_{A_j} - P_{i(j)}')x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$. We may find a central projection Q_j majorized by $P_{i(j)}'$ with

$$||(P_{i(j)}' - Q_j)x|| \leqq \epsilon/3n$$

for every $x \in \mathfrak{X}$ such that

$$A_j P Q_j = B_j Q_j$$

for some $B_j \in \mathfrak{A}$ (Lemma 1). Since all $C_{B_j} P_{i(j)}'$ are in $\langle \mathfrak{P}_0 \rangle$, their least upper bound $Q = \sum C_{B_j} P_{i(j)}'$ is also in $\langle \mathfrak{P}_0 \rangle$. We then have that

$$\begin{aligned} ||(P - Q)x|| \leqq ||P(1 - \textstyle\sum_j P_{i(j)}')x|| + \textstyle\sum_j ||P(P_{i(j)}' - C_{A_j})x|| + \\ \textstyle\sum_j ||(C_{A_j}P - C_{B_j})P_{i(j)}'|| \, ||(P_{i(j)}' - Q_j)x|| \leqq \epsilon \end{aligned}$$

for every $x \in \mathfrak{X}$.

Now we consider the corollary mentioned in the introduction. We remind the reader that an automorphism α of $\mathfrak{A}$ is said to be ideal-preserving if $\alpha(\mathfrak{J}) \subset \mathfrak{J}$ for each ideal $\mathfrak{J}$ of $\mathfrak{A}$. For GCR algebras this is equivalent to the requirement that $\alpha(\mathfrak{J}) = \mathfrak{J}$ [**7**]. Further, the automorphism α is said to be universally weakly inner if, for every representation π of $\mathfrak{A}$, the automorphism α_π of $\pi(\mathfrak{A})$ defined by $\alpha_\pi(\pi(A)) = \pi(\alpha(A))$ is implemented by a unitary element of $\pi(\mathfrak{A})^-$. (Note that α_π is well-defined whenever α is ideal-preserving.) Lance [**7**] showed that ideal-preserving automorphism of a separable GCR algebra is universally weakly inner. In a recent paper Elliott [**5**] proved that same result for an arbitrary GCR algebra by using a decomposition into "champs continus". At the end of the paper he asked if the fact that the enveloping von Neumann algebra of a GCR algebra is type I could be used to prove the same result. The following corollary answers this question in the affirmative.

COROLLARY 4. *An ideal preserving automorphism α of a GCR algebra $\mathfrak{A}$ is universally weakly inner.*

Proof. We identify $\mathfrak{A}$ with a weakly dense *-subalgebra of the enveloping von Neumann algebra $\mathfrak{B}$ of $\mathfrak{A}$ and prove that α is inner in $\mathfrak{B}$. The automorphism α has an extension to an automorphism of $\mathfrak{B}$. We also denote this extension by α. Since $\mathfrak{A}$ is GCR, the algebra $\mathfrak{B}$ is of type I and so it suffices to show that $\alpha(P) = P$ for every $P \in \mathfrak{P}$ [**4**, III, § 3, Corollary 2 of Proposition 3]. Clearly, if P is open, we have that $\alpha(P) = P$ by the strong continuity and ideal preserving property of α and α^{-1}. Thus the same relation holds for finite Boolean polynomials in open projections and hence for strong limits of such polynomials. By Theorem 2, every central projection of $\mathfrak{B}$ is obtained in this way and so there is a unitary $U \in$ such that $\alpha(A) = UAU^{-1}$ for every A in $\mathfrak{B}$.

Now let π be an arbitrary nondegenerate representation of $\mathfrak{A}$. There is a normal homomorphism ϕ of $\mathfrak{B}$ onto the von Neumann algebra $\pi(\mathfrak{A})''$ generated by $\pi(\mathfrak{A})$ such that $\phi(A) = \pi(A)$ for every $A \in \mathfrak{A}$ [**3**, 12.1.5]. Then we have that

$$\begin{aligned}\alpha_\pi(\pi(A)) &= \pi(\alpha(A)) = \phi(\alpha(A))\\ &= \phi(UAU^*) = \phi(U)\phi(A)\phi(U)^*\\ &= \phi(U)\pi(A)\phi(U)^*\end{aligned}$$

for every $A \in \mathfrak{A}$. Clearly, the operator $\phi(U)$ is unitary in $\pi(\mathfrak{A})''$. Hence the automorphism α_π is implemented by the unitary operator $\phi(U)$ in

$$\pi(\mathfrak{A})^- = \pi(\mathfrak{A})''.$$

Addendum (July 13, 1972). Trond Digernes has shown that a separable GCR algebra is completely characterized by the fact that $\mathfrak{P}_0 = \mathfrak{P}$. This will appear in an article, *A new characterization of separable GCR-algebras* in Proc. Amer. Math. Soc. Herbert Halpern has recently shown that a C^*-algebra (perhaps non-separable) is GCR if and only if the Boolean σ-algebra generated by $\mathfrak{P}_0$ contains all minimal projections of $\mathfrak{P}$.

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RIGHT QUOTIENT RINGS OF A RIGHT LCM DOMAIN

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In this paper we continue our investigation of the class of right LCM domains which was introduced in [**2**]. A right LCM domain is an (not necessarily commutative) integral domain with unity in which the intersection of any two principal right ideals is again principal. In this note we study the right quotient rings of such a ring. In Section 1 we describe some of the characteristic properties of right quotient monoids with respect to which quotient rings are formed. Three particular types of quotient rings are described in Section 2. In Section 3 we relate the right ideals of a ring to those of its quotient ring. The right D-chain which is constructed for a principal right ideal domain in [**1**] is described in a more general context in Section 4. Each ring below is assumed to be a ring with unity having no proper divisors of zero.

1. Right quotient monoids. Let S be a submonoid of the monoid R^* of nonzero elements of a ring R. It is well known that if

$$aR \cap bS \neq \emptyset \text{ for each } a \in S, b \in R \tag{1}$$

then $K = RS^{-1} = \{ba^{-1} | b \in R, a \in S\}$ is a ring under operations that extend those of R. Addition and multiplication are carried out in K by using the fact that for each $a \in S$ and $b \in R$, $a^{-1}b = b'(a')^{-1}$ for $ab' = ba' \in aR \cap bS$ (see [**4**]).

A submonoid S of R^* satisfying (1) is a *right Ore system in R*. If R^* is a right Ore system in R then R is a *right Ore domain*. A submonoid S of R^* is *saturated* if for all $a, b \in R$,

$$ab \in S \text{ implies } a \in S \text{ and } b \in S.$$

We shall deal mainly with saturated right Ore systems; these are called *right quotient monoids in R*. If S is a right Ore system in R with $K = RS^{-1}$ then S is contained in the group of units U_K of K; furthermore, $U_K \cap R = S$ if and only if S is saturated. For example, the group of units U_R of R is a right quotient monoid in R. If R is a right Ore domain then R^* is a right quotient monoid in R and $K = R(R^*)^{-1}$ is the right quotient field of R.

For two nonzero elements a, b in a ring R the greatest common left divisor (gcld) of a and b is denoted by $(a, b)_l$ and the least common right multiple (lcrm) is denoted by $[a, b]_r$; $(a, b)_r$ and $[a, b]_l$ have analogous meanings. If two elements $a, b \in R$ have a nonzero common right multiple, say $ab' = ba' \neq 0$, and if $[a, b]_r$ exists, then it is easy to show that $(a', b')_r$ also exists and satisfies

$$ab' = ba' = [a, b]_r(a', b')_r. \tag{2}$$

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This is the case, for example, in a right LCM domain where $[a, b]_r$ is a generator of $aR \cap bR$ if this intersection is nonzero. Similar remarks hold with "right" replaced by "left"; the analogue of (2) is

$$ab' = ba' = (a, b)_l[a', b']_l$$

(see [**2**, Lemma 1]). If S is a right quotient monoid in R and if $a, b \in S$ have a lcrm then it is also a member of S since we have a relation (2) with $a' \in S$. Of course, $(a, b)_l \in S$ if it exists. Similarly the lclm and gcrd of two elements in S are again in S.

For two elements $a, a' \in R$ we define $a \operatorname{tr} a'$ if there is a relation $ab' = ba'$ which is both left and right coprime, that is, for which $(a, b)_l = (a', b')_r = 1$. In view of (2) we have $a \operatorname{tr} a'$ if and only if there exist $a', b' \in R$ such that $ab' = ba' = [a, b]_r$ and $(a, b)_l = 1$; if we require, in addition, that $(a, b)_l$ be a (right) linear combination of a and b then a and a' are similar (see [**4**]).

A submonoid S of R^* is said to be tr-*closed* if whenever $a \operatorname{tr} a'$ and $a \in S$ then $a' \in S$; if $a \in S$ if and only if $a' \in S$ we say that S is rtr-*closed*. The following theorem gives a useful characterization of right quotient monoids for a large class of right LCM domains.

THEOREM 1. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf. *Let S be a saturated submonoid of R^*. Then S is a right quotient monoid in R if and only if*

(i) *each element a of S is (right) large, that is, $aR \cap bR \neq 0$ for each $b \in R^*$,*

(ii) *S is* tr-*closed.*

Proof. If S is a right quotient monoid in R then each element of S is large because of (1). Suppose $a \operatorname{tr} a'$ so that $ab' = ba' = [a, b]_r$ with $(a, b)_l = 1$. Choose $a_1 \in S$ and $b_1 \in R$ such that $ab_1 = ba_1$. Thus $a_1 = a'x$ for some $x \in R$, and a' is then in S since S is saturated. Conversely, if S is a saturated submonoid of R^* satisfying (i) and (ii) then for $a \in S$ and $b \in R$ we may choose $0 \neq ab' = ba' \in aR \cap bR$. If $(a, b)_l = d$ and $[a, b]_r = m$ then $a = da_1$, $b = db_1$, $a_1b' = b_1a' = [a_1, b_1]_r$, and $(a_1, b_1)_l = 1$. Consequently $a_1 \operatorname{tr} a'$; moreover $a_1 \in S$ and therefore $a' \in S$ and, this establishes (1).

2. Three particular types of right quotient rings.

We begin this section with a general theorem that applies to all right LCM domains. Recall that an element a of a ring R is *(right) large* if aR is a large right ideal, that is, $aR \cap xR \neq 0$ for each $x \in R^*$.

THEOREM 2. *Let R be a right* LCM *domain.*

(i) *The set L_R of all elements of R that are (right) large is a right quotient monoid in R.*

(ii) *R has a right quotient ring K such that $L_K = U_K$, namely, $K = R(L_R)^{-1}$.*

Proof. The set L_R is not empty since each unit is large. If $ab \in L_R$ then $a \in L_R$ since $abR \subset aR$. Also, for any $x \in R^*$ we have $abR \cap axR \neq 0$ which shows

$b \in L_R$. On the other hand if both a, $b \in L_R$ and $x \in R^*$ then we have $ax' = xa' \neq 0$, $bx'' = x'b' \neq 0$ and therefore $abx'' = x(a'b') \neq 0$ so that $ab \in L_R$. Finally, let $a \in L_R$, $b \in R$ with $[a, b]_r = ab' = ba'$. For any $x \in R^*$, $aR \cap bR \supset aR \cap bxR \neq 0$ so that $aR \cap bR \cap bxR \neq 0$, that is, $ba'R \cap bxR \neq 0$ and this shows $a' \in L_R$. Thus L_R is a right quotient monoid in R. To prove the second part let $z \in L_K$ where $K = R(L_R)^{-1}$; we assume, without loss in generality, that $z \in R$. For any $x \in R^*$ we have $zK \cap xK \neq 0$, say $zb_1a_1^{-1} = xb_2a_2^{-1}$. If $a_1a_2' = a_2a_1'$ (in L_R) then $zb_1a_2' = xb_2a_1'$ so that z is also large in R. Consequently z is a unit in K.

We turn to right LCM domains R that satisfy the following condition:

(3) $[x, yz]_r = [x, y]_r$, $(x, yx)_l = (x, y)_l$ implies z is a unit.

It is shown in [**2**, Theorem 2] that for such rings, the factorization of an element into primes (nonunits that have no proper factors) is unique up to order of factors and projective factors. In general, two elements a, $a' \in R$ are *(right) projective* (a pr a') if there is a sequence $a = a_1, a_2, \ldots, a_n = a'$ in R such that either a_i tr a_{i+1} or a_{i+1} tr a_i for each i. In a ring R, the *(right) dimension*, $\dim(a)$, of a may be defined to be the supremum of the lengths of all maximal chains in the poset $\mathscr{L}(aR)$ of all principal right ideals of R containing aR. In a right LCM domain that satisfies (3), $\dim(a)$ is just the number of prime factors in any factorization of a if a is the product of primes, and ∞ otherwise.

If R is a right LCM domain in which each pair of elements having a nonzero common right multiple has a hclf then for each $z \in R^*$ the poset $\mathscr{L}(zR)$ is a lattice; the sup of aR and bR in $\mathscr{L}(zR)$ is $(a, b)_lR$ and the inf of aR and bR is the intersection $[a, b]_rR$ of aR and bR. If, in addition, R satisfies (3) then it is easy to check that each lattice $\mathscr{L}(zR)(z \in R^*)$ is modular; in fact, (3) is equivalent to the modular law [**2**, Theorem 1]. In such a ring, if a tr a' then the modular lattices $\mathscr{L}(aR)$ and $\mathscr{L}(a'R)$ are transposes and consequently isomorphic; in particular, $\dim(a) = \dim(a')$. Our remarks together with Theorem 2 indicate that the set

$$R' = \{a \in L_R | \dim(a) < \infty\}$$

is a saturated submonoid of L_R which is tr-closed, and rtr-closed if each element of R^* is large. We summarize with the following

THEOREM 3. *Let R be a right* LCM *domain satisfying* (3) *in which each pair of elements having a nonzero common right multiple has a* hclf. *The set R' of all large finite-dimensional elements of R is a right quotient monoid in R which is* rtr*-closed if R is a right Ore domain.*

The right quotient ring associated with R' in Theorem 3 will be considered more closely in Section 4.

Let R be a ring satisfying the hypotheses in Theorem 3. For a set I of primes in R, let S_I be the set of all elements of R' whose prime factors are projective to primes in I. In particular, if p is a prime in R and if I is the set of all primes in R *not* projective to p, we write S_p for S_I. Now S_I is a saturated submonoid of R'

because of the uniqueness of prime factorizations. In addition, if $a \operatorname{tr} a'$ and $a = p_1 \ldots p_n$ is a prime factorization of a then there exist primes $p_1', \ldots, p_n'$ in R such that $a' = p_1' \ldots p_n'$ and $p_i \operatorname{tr} p_i'$; this follows immediately from [**2**, Theorem 5 (part i)]. Therefore S_I is tr-closed. We have established the following

THEOREM 4. *Let R be a right* LCM *domain satisfying* (3) *in which each pair of elements having a nonzero common right multiple has a* hclf. *If I is a set of primes in R then S_I is a right quotient monoid in R. In particular, for any prime p in R, R has a right quotient ring at p, namely $K_p = R(S_p)^{-1}$.*

We remark that our results apply to the free associative algebra $R = D[X]$ over a commutative unique factorization domain D. Each $\mathscr{L}(zR)$ (for $z \in R^*$) is a finite-dimensional modular lattice [**3**, Theorem 8] so that R satisfies the hypotheses of Theorem 4. In this example we have $R' = L_R$.

3. The closure of a right ideal. Let S be a right quotient monoid in a ring R with $K = RS^{-1}$. If A is a right ideal of R then $AS^{-1} = \{as^{-1} | a \in A, s \in S\}$ is a right ideal of K and $AS^{-1} \cap R$ is a right ideal of R containing A. We define the *S-closure* of A by $\mathrm{cl}(A) = AS^{-1} \cap R$. A right ideal A of R is *S-closed* if $\mathrm{cl}(A) = A$. It is not difficult to prove that the poset of closed right ideals of R is a lattice which is isomorphic to the lattice of all right ideals of K; the isomorphism is given by $A \to AS^{-1}$ with the inverse given by $B \to B \cap R$.

An element $b \in R$ is *S-closed* if bR is a closed right ideal. In this case b is *right prime to S*, that is, b has no nonunit right factor that belongs to S. For, if $b = b_1a_1$ with $a_1 \in S$ then since a_1 is a unit in K we have $\mathrm{cl}(bR) = bK \cap R = b_1K \cap R = \mathrm{cl}(b_1R)$ from which we deduce $bR = b_1R$. Thus a is a unit in R. The converse, which may be established in a special case, is included in the following.

THEOREM 5. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf. *Let S be a right quotient monoid in R which is* rtr-*closed and let $K = RS^{-1}$.*

(i) *An element $b \in R$ is S-closed if and only if b is right prime to S. Furthermore, if R has the* acc *for principal right ideals then*

(ii) *the S-closure of each principal right idea of R is also a principal right ideal,*

(iii) *each element $z \in R$ has a unique factorization $z = ba$ where $a \in S$ and b is right prime to S.*

Proof. Let $b \in R$ be right prime to S. Now $bR \subset \mathrm{cl}(bR)$; to establish the reverse inclusion let $z = brs^{-1} \in \mathrm{cl}(bR)$ with $m = [z, b]_r$, $d = (z, b)_l$. Then $z = dz_1, b = db_1, m = dm_1$ and $(z_1, b_1)_l = 1, [z_1, b_1]_r = m_1$. If $m_1 = z_1b_1' = b_1z_1'$ then $b_1 \operatorname{tr} b_1'$. Since $zs = br$ we have $zs = mx$ for some $x \in R$. Thus $s = b_1'x$ and $b_1' \in S$ since S is saturated; consequently $b_1 \in S$ because S is rtr-closed. Since b is right prime to S, b_1 must be a unit. Therefore $b = (z, b)_l$ and $z \in bR$ as desired. This shows that if b is right prime to S then b is S-closed. The converse follows by our remarks preceding the theorem.

For each $z \in R$ let $D_z = \{bR | z = ba \text{ for some } a \in S\}$. Using the acc for principal right ideals we may choose bR maximal in D_z. Thus $z = ba$ where $a \in S$ and b is right prime to S. Since $a \in S$, $zK = bK$ so that $\mathrm{cl}(zR) = \mathrm{cl}(bR) = bR$. This establishes (ii); the third part follows from the first two.

Let R be a right LCM domain such that each pair of elements having a nonzero common right multiple has a hclf. If S is a saturated right quotient monoid in R then the poset $\mathscr{L}(zR)'$ of closed right ideals in the lattice $\mathscr{L}(zR)$ is a sublattice of $\mathscr{L}(zR)$. The mapping $\varphi : \mathscr{L}(zR)' \to \mathscr{L}(zK)$ defined by $\varphi(bR) = bK$ is easily seen to be an order preserving injection. Also, if $bK \in \mathscr{L}(zK)$, then under the hypotheses of Theorem 5 we may assume b is right prime to S so that bR is S-closed; thus φ is also a surjection in this case. We summarize our remarks as follows.

THEOREM 6. *Let R be a right* LCM *domain in which each pair of elements having a nonzero common right multiple has a* hclf, *and assume R has the* acc *for principal right ideals. Let S be a right quotient monoid in R which is* rtr-*closed and let $K = RS^{-1}$. The poset $\mathscr{L}(zK)$ of principal right ideals of K containing zK and the lattice $\mathscr{L}(zR)'$ of closed principal right ideals of R containing zR are lattice isomorphic. In particular, K is also a right* LCM *domain satisfying the* acc *for principal right ideals in which each pair of elements having a nonzero common right multiple has a* hclf.

4. The right *D*-chain. Throughout this section R will denote a ring satisfying the acc for principal right ideals such that each pair of nonzero elements has a lcrm and hclf satisfying (3). Such a ring may be characterized as a right Ore right LCM domain R such that for each $z \in R^*$, $\mathscr{L}(zR)$ is a modular lattice satisfying the acc. Thus if S is a right quotient monoid in R which is rtr-closed then $K = RS^{-1}$ is a ring of the same kind as R by Theorem 6.

Since each element of R^* is large we see by Theorem 3 that the set R' of finite dimensional elements of R is a right quotient monoid in R which is rtr-closed. Since $K = R(R')^{-1}$ must be a ring of the same kind as R we may consider K' and iterate this procedure. This leads to the right D-chain for R which we shall describe presently. The construction which was first described in [**1**] for the particular case of a principal right ideal domain depends on the following.

LEMMA. *If $K = R(R')^{-1}$ then $K' \cap R$ is a right quotient monoid in R which is* rtr-*closed.*

Proof. Clearly $ab \in K' \cap R$ if and only if $a, b \in K' \cap R$. If $a \operatorname{tr} a'$ in R then $a \operatorname{tr} a'$ in K; for if $aR \cap bR = ba'R$ and $(a, b)_l = 1$ then clearly $ba'K = (aR \cap bR)S^{-1} = aK \cap bK$. Suppose $aK \subset dK$ and $bK \subset dK$ where $d \in R$. Since R has the acc for principal right ideals we may assume d is right prime to S. Thus $aR \subset aK \cap R \subset dK \cap R = dR$; similarly $bR \subset dR$ so that d is a unit since $(a, b)_l = 1$ in R. Thus $(a, b)_l = 1$ in K and $a \operatorname{tr} a'$ in K. Therefore $K' \cap R$ is rtr-closed and the conclusion of the lemma follows from Theorem 1.

Using transfinite induction we construct the *right D-chain* $\{R^{(\alpha)} | 0 \leqq \alpha \leqq \delta\}$ of right quotient monoids in R together with the associated chain $\{K_\alpha = R(R^{(\alpha)})^{-1} | 0 \leqq \alpha \leqq \delta\}$ of right quotient rings of R as follows. Let $R^{(0)} = U_R$ and $K_0 = R$. Let $\alpha > 0$ be any ordinal; if $\beta < \alpha$ assume $R^{(\beta)}$ has been defined and is a right quotient monoid in R which is rtr-closed and let $K_\beta = R(R^{(\beta)})^{-1}$. We define $R^{(\alpha)}$ by

$$R^{(\alpha)} = \begin{cases} (K_{\alpha-1})' \cap R, \text{ if } \alpha \text{ is a nonlimit ordinal} \\ \bigcup_{\beta<\alpha} R^{(\beta)}, \text{ if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Our induction is valid by the lemma. Clearly $R^{(\beta)} \subset R^{(\alpha)}$ and $K_\beta \subset K_\alpha$ if $\beta \leqq \alpha$. Also $R^{(\alpha)} = R^{(\alpha+1)}$ for some ordinal α; if δ is the least such ordinal then $R^{(\delta)} = R^*$ so that K_δ is the right quotient field of R.

One of the important uses of the right D-chain is in describing the unique factorization property in R to which we now turn. By using Theorem 5 (iii) and finite induction we see that each element $a \in R^*$ may be written uniquely (up to unit factors) as

$$a = z_{\alpha_1} z_{\alpha_2} \ldots z_{\alpha_k} u$$

where α_i are nonlimit ordinals such that $\alpha_0 \geqq \alpha_1 > \ldots > \alpha_k$, $z_{\alpha_i} \in R^{(\alpha_i)}$ and is right prime to $R^{(\alpha_i-1)}$, and u is a unit in R (see [**1**, Theorem 2]). Each z_{α_i} may be characterized as follows. An element $x \in R^*$ is an *α-prime* if xR is maximal in $\{xR | x \in R^{(\alpha)} \backslash R^{(\alpha-1)}\}$ ($\alpha > 0$ a nonlimit ordinal). Note that 1-primes are the usual primes in R; if $\alpha > 1$, α-primes have infinite dimension in R and are prime in $K_{\alpha-1}$. Each z_{α_i} above is a product of α_i-primes which is unique up to order of factors and projective factors in K_{α_i-1}. Since the proofs are substantially the same as those in [**1**] we shall not repeat them here. We summarize in the following

THEOREM 7. *Let R be a ring satisfying the* acc *for principal right ideals in which each pair of nonzero elements has a* lcrm *and* hclf *satisfying* (3). *Let $\{R^{(\alpha)} | 0 \leqq \alpha \leqq \delta\}$ be the right D-chain in R with $\{K_\alpha | 0 \leqq \alpha \leqq \delta\}$ the associated chain of right quotient rings of R. Each element $a \in R^*$ has a unique (up to unit factors) factorization $a = z_{\alpha_1} \ldots z_{\alpha_k} u$ where $\alpha_0 \geqq \alpha_1 > \ldots > \alpha_k$, each z_{α_i} is a product of α_i-primes in R (which is unique up to order of factors and projective factors in K_{α_i-1}) and u is a unit in R.*

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THE CONTINUITY OF THE VISIBILITY FUNCTION ON A STARSHAPED SET

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1. Introduction.

Definition. The visibility function assigns to each point x of a fixed measurable set E in a Euclidean space E_n the Lebesgue outer measure of $S(x)$, the set $\{y : rx + (1 - r)y \in E$ for every r in $[0, 1]\}$.

The purpose of this paper is to determine sufficient conditions for the continuity of the function on the interor of a starshaped set.

2. Preliminaries. We basically use the same terminology as in [**1**], where the reader may find a more general investigation of the continuity properties of the visibility function. Lebesgue measure in E_n is denoted by m or m_n (if more than 1 measure is under discussion). The convex kernel of E, $\{x \in E : S(x) = E\}$ is expressed as conv ker E, and the convex hull of E is denoted by conv E. The open r-ball about a point x is given by $B_r(x)$. The interior of E relative to the smallest flat containing E is given by intv E. Finally, xy will denote the line segment joining x to y, $L(x, y)$ will denote the line determined by x and y, and $\langle W \rangle$ will denote the flat generated by the set of vectors W.

In the sequel, we must draw upon 3 facts established in [**1**], which we state as theorems. As in [**1**], we will designate the visibility function for a fixed set by v.

THEOREM 1. *If $O \subset E_n$ is open, then v is lower semicontinuous on O.*

THEOREM 2. *If $K \subset E_n$ is compact, then v is upper semicontinuous on K.*

THEOREM 3. *Let E be a compact set in E_n. If $x \in E$, the set of endpoints of all maximal segments in $S(x)$ with one endpoint x forms a measurable set and has measure zero.*

It is easy to see that the visibility function may be discontinuous on the interior of a compact starshaped set in E_n, if the dimension of the convex kernel does not exceed $n - 2$. For example, let K be a Cantor set of positive measure in $[0, 2\pi]$ and let E be the following planar starshaped set: $\{(r, \theta) : r \leqq 1\} \cup \{(r, \theta) : 1 < r \leqq 2, \theta \in K\}$. Let $q \in E \cap \{(r, \theta) : r \leqq 1\}$. Since $E \cap \{(r, \theta) : 1 < r \leqq 2\}$ is nowhere dense, and E is starshaped with respect to 0, $S(q) \cap E \cap \{(r, \theta) : 1 < r \leqq 2\} \subset L(q, 0)$, so that the visibility function for E is discontinuous at the origin. Using "Cantor cylinders", we may construct analogous examples in E_n for any n.

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3. Results. In establishing our main theorem, we use induction and a basic property of generalized cylindrical coordinates. Specifically given any flat F of dimension $n - 2$ in E_n, we can find a collection of hyperplanes $\{H_\theta\}$, $\theta \in [0, \pi)$, such that $F = H_{\theta_1} \cap H_{\theta_2} (\theta_1 \neq \theta_2)$, $\bigcup H_\theta = E_n$, and if K is an arbitrary Borel set satisfying $m_{n-1}(H_\theta \cap K) = 0$ for almost every θ, then $m(K) = 0$.

THEOREM 4. *Let E be a compact starshaped set in E_n such that* int $E \neq \emptyset$. *Suppose* dim conv ker $E \geqq n - 1$. *Then the visibility function v is continuous on* int E.

Proof. We first establish our theorem in the case $n = 2$. Let x be an arbitrary point of int E different from some point in conv ker E and let $\{x_n\} \to x$. The lower semicontinuity of v at x follows if we can show $S = \{y : y \in S(x), y \notin \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)\}$ has measure zero. If we denote the set of points which x sees via E but not via int E by M, then clearly $S \subset M$. First, it can be seen that any ray with endpoint x intersects M in one point or an interval. Excluding the ray on the one possible line which might contain all the points of conv ker E, if such a ray R contains an interval in M, we associate a rational point in E_2 with it. Fix $p \in$ conv ker E, $p \neq x$. $L(p, x)$ divides the plane into two open half planes, H_1 and H_2. Suppose without loss of generality $R \subset H_1$ and $y_1y_2 \subset M \cap R$. Clearly there exists a point z such that $z \in$ intv y_1y_2 and pz passes through a point r_R in H_1 with rational coordinates. We claim the assignment $R \to r_R$ is 1-1. Suppose there were another ray R' such that $R' \subset H_1$ and R' were also assigned r_R. Then there exists z' on $R' \cap M$ such that r_R is in intv pz' and we may harmlessly suppose $z \in$ intv $z'p$. Since conv$(p \cup z' \cup x) \subset E$, it follows that $z \notin M$, a contradiction.

The remaining points of M not contained in these intervals must be endpoints of maximal segments in $S(x)$ with one endpoint x. But these points have measure zero by Theorem 3. Hence, $m(S(x)) \leqq m(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) \leqq \liminf m(S(x_k))$ and the lower semicontinuity of v at x follows in the case $n = 2$.

For general n, we must distinguish two cases for an arbitrary point $x \in$ int E:

(1) there exists n independent points $\{y_1, \ldots, y_n\} \subset$ conv ker E such that $x \notin \langle y_1, \ldots, y_n \rangle$, and

(2) there exists n independent points $\{y_1, y_2, \ldots, y_n\} \subset$ conv ker E such that $x \in \langle y_1, \ldots, y_n \rangle$.

If dim conv ker $E = n$, both conditions are satisfied for every x, and if dim conv ker $E = n - 1$, then exactly one is satisfied by each x in E. (See Valentine [3] for a thorough discussion of flats, convex kernels and convex hulls).

In case (1) we first establish by induction that if p is any point in intv conv$(\{y_1, \ldots, y_n\})$ where $\{y_1, \ldots, y_n\}$ are as above, then if $\{x_n\} \to x$ on $L(x, p)$ we have $m(S(x)/\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) = 0$. This, of course, has been shown when $n = 2$. Assume it is true if $n = k$, and now suppose $n = k + 1$. Let $\{y_1, \ldots, y_{k+1}\}$ be independent points in conv ker E satisfying $x \notin \langle y_1, \ldots, y_{k+1} \rangle$. Let $p \in$ intv conv$(\{y_1, \ldots, y_{k+1}\})$. Clearly there exists a set of hyperplanes

$\{H_\theta\}$, $\theta \in [0, \pi)$, as in the previous discussion such that $L(x, p) \subset H_\theta$ for every θ, dim conv ker $(H_\theta \cap E) \geqq k - 1$, $\bigcup H_\theta = E_{k+1}$ and for every θ, there exists independent points $\{y_1^\theta, \ldots, y_k^\theta\}$ contained in conv $\ker(H_\theta \cap E)$ such that $p \in$ intv conv$(\{y_1^\theta, \ldots, y_k^\theta\})$ and $x \notin \langle y_1^\theta, \ldots, y_k^\theta \rangle$.

Now let $\{x_n\}$ be an arbitrary sequence of points on $L(x, p)$ converging to x. By the induction hypothesis we have $m_k(S(x) \cap H_\theta / \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n) \cap H_\theta) = 0$. Hence, by our previous remarks we have $m_{k+1}(S(x)/\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} S(x_n)) = 0$. Hence our proposition is true in E_n for every n.

This all of course implies that given any point $p \in$ intv conv$(\{y_1, \ldots, y_n\})$, v is continuous on $L(x, p)$. Therefore there exists a point x_0 in E such that $x \in$ intv px_0 and $v(x_0) > v(x) - \epsilon$. Since p was chosen in intv conv$(\{y_1, \ldots, y_n\})$, it follows that conv$\{x_0, y_1, \ldots, y_n\}$ will contain a neighborhood N of x and since $y_i \in$ conv ker E, $i = 1, 2, \ldots, n$, we conclude that $v(y) > v(x) - \epsilon$ for every $y \in N$ so that v is lower semicontinuous at x.

In case (2) we establish by induction that the set M of points which x sees via E but not via int E has measure zero, which is enough to establish the continuity of the visibility function at such points as we have noted before. We have seen this to be true when x is any interior point of E if $n = 2$. Now suppose the assertion has been established for $n = k$. If $n = k + 1$ we again rotate a hyperplane to sweep out $k + 1$ space such that at each stage H_θ, $\theta \in [0, \pi)$, $E \cap H_\theta$ satisfies the induction hypothesis. Let H denote a hyperplane containing x and a subset of conv ker(E) of dimension k. There exists a flat $F \subset H$, dim $F = k - 1$, such that $x \in F$ and dim conv ker$(F \cap E) = k - 1$. Let $\{H_\theta\}$, $\theta \in [0, \pi)$, denote the set of hyperplanes generated by rotating $H = H_0$ about F. Then for all $\theta \in [0, \pi)$, we have dim conv ker$(H_\theta \cap E) \geqq k - 1$, and x is located on a hyperplane in H_θ (namely F) for every θ containing a subset of conv ker$(H_\theta \cap E)$ of dimension $k - 1$. By the induction hypothesis the set $M_\theta =$ those points of $E \cap H_\theta$ which x sees via $E \cap H_\theta$ but not via intv $E \cap H_\theta$ has k dimensional measure zero. By our earlier remarks, $\bigcup_{\theta \in [0,\pi)} M_\theta$ has $k + 1$ dimensional measure zero.

We claim that $M/M \cap H_0 \subset \bigcup_{\theta \in [0,\pi)} M_\theta$. Suppose z is an interior point of $E \cap H_\theta / F$ relative to H_θ where $\theta \neq 0$. Let N be an H_θ neighborhood of z contained in $E \cap H_\theta/F$. Then $\dim(N \cup \text{conv ker } E) = \text{k} + 1$, and $z \in$ int conv$(N \cup$ conv ker $E) \subset$ int E. Hence boundary points of E on H_θ, $\theta \neq 0$, are boundary points of $E \cap H_\theta$ relative to H_θ. Thus, $M/M \cap H_\theta \subset \bigcup M_\theta$ so that $m(M) = 0$, and the continuity of v at such points x follows.

Some observations are now in order. Clearly, the converse of Theorem 4 fails. If E is a compact starshaped set in E_n the dimension of whose convex kernel exceeds $n - 2$, then the boundary of E has measure zero. Thus, the reader might guess that Theorem 4 is a special case of the following more general theorem: if $E \subset E_n$ is a compact set whose boundary is of measure zero, then the visibility function is continuous on int E. However, the above proposition is false. In the standard Cantor set of measure π in $[0, 2\pi]$ derived by tossing out a sequence of open sets $\{O_n\}$ from $[0, 2\pi]$ in the usual way, let our residual closed set after the nth deletion be called K_n. Letting

$$r_n = 3 \sup_{x \in K_n} \inf_{y \in [0,2\pi]/K_n} |y - x|,$$

a counterexample is seen to be

$$\{(r, \theta) : r \leqq 2\} / \bigcup_{n=1}^{\infty} \{(r, \theta) : \theta \in O_n, 1 < r < 1 + \sqrt{r_n}\}.$$

For details, see [**2**].

For the case when E is a bounded open starshaped set, we are only able to give the following planar result.

THEOREM 5. *Let O be any bounded open starshaped set in the plane. Then the visibility function is continuous on O.*

Proof. Let x be an arbitrary element of O and let $\{x_n\} \to x$. If $x \in \operatorname{conv} \ker O$, $v(x_n) \to v(x)$, so we may assume $x \notin \operatorname{conv} \ker O$. We show that $S = \{y : y \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} S(x_n)/S(x)\}$ has measure zero. Fix a point p in conv ker O and consider any ray R emanating from x. We claim that $R \cap S$ is either empty or contains a line segment. If the line L determined by R contains p, then $R \cap S = \emptyset$. If not, and $R \cap S \neq \emptyset$, then all but finitely many of the $\{x_n\}$ which see a fixed point y of $R \cap S$ must lie on the p side of L, or else we have $\operatorname{conv}(x \cup y \cup p) \subset O$. Since $\operatorname{int} \operatorname{conv}(y \cup p \cup x) \subset O$, there exists an open rectangle in O with one edge xv containing y in its relative interior where $yv \subset O$. It is clear that all but finitely many members of the range of $\{x_n\}$ which could see y can also see yv, and since $yv \cap S(x) = \emptyset$, $R \cap S$ contains yv, an interval.

We now proceed in the same manner as in the compact case: to each ray R containing an interval in S we associate a rational point r_R.

This point corresponds uniquely to R, for suppose that r_R lies on both pw and pw' where $w \in R \cap S$, $w' \in R' \cap S$ and $w \in \operatorname{intv} w'p$, say. Since $\operatorname{int} \operatorname{conv}(w' \cup x \cup p) \subset 0$, we have $w \in S(x)$, a contradiction. The upper semicontinuity of v now follows in the obvious way.

In addition to establishing more general results for open sets, the following conjecture is of interest: let E be a compact starshaped set in E_n whose convex kernel is of dimension $n - 2$. If $x \in \operatorname{int} E$ is a point of discontinuity of the visibility function, then x is a point on the smallest flat containing the convex kernel of E.

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